I. FLUID FLOW IN A PRECESSING SPHERICAL CAVITY

II. ELECTROMAGNETIC RADIATION FROM AN EXPANDING SPHERE IN A MAGNETIC FIELD

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ABSTRACT

I. FLUID FLOW IN A PRECESSING SPHERICAL CAVITY
II. ELECTROMAGNETIC RADIATION FROM AN EXPANDING SPHERE IN A MAGNETIC FIELD

by Giulio Venezian

In Part I the flow of an incompressible fluid inside a precessing spherical cavity is studied. The precession angle is assumed small and the equations of motion are linearized. For the case of large viscosity an expansion is developed in inverse powers of the viscosity by expanding the velocity field in vector spherical harmonics. The flow obtained is essentially rigid body motion. The case of low viscosity is also studied. At low precession rates difficulties arise in the boundary layer treatment and the inviscid equations. A modified boundary layer equation is derived and an approximate solution obtained. The flow consists essentially of rotation about the average axis of rotation. Some geophysical aspects of the problem, and in particular its relevance to dynamo theories of the earth's magnetic field are discussed.

Part II deals with the electromagnetic fields about a perfectly conducting sphere which is placed in a uniform magnetic field. The radiation fields that result when the radius of the sphere is allowed to change are investigated. Explicit expressions are obtained for the cases of a sphere expanding or collapsing at a uniform rate. In the latter case it is found that wave propagation and energy propagation are in opposite directions. Constant speed oscillations are also investigated and the effect of the amplitude on the power radiated is considered. The case of arbitrary motions of the radius is also discussed.

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I. FLUID FLOW IN A PRECESSING SPHERICAL CAVITY

1. INTRODUCTION

The investigation described here deals with the flow of a viscous fluid inside a precessing spherical container. This problem arises in the study of the effect of the earth's precession on its liquid core, and is of particular significance in connection with the dynamo theory of the earth's magnetic field. To indicate the position that this problem occupies in the general geophysical picture, a brief review of the pertinent aspects relating to the terrestrial field is given in this chapter. This review includes a description of the earth's magnetic field and an outline of the evolution of the theories that have been advanced to explain it.

1.1 The Earth's Magnetic Field[1]

The earth's magnetic field has been studied through observations taken outside its surface, principally near the surface itself. From a detailed knowledge of the field at the surface it is possible to deduce the field outside it but no information can be obtained as to its character in the interior. For example, the external field can roughly be described as that which would exist if the earth had a dipole at its center; but the same external field would result from a uniformly

[1] This discussion is based on a review article by E. C. Bullard, "The Interior of the Earth" in The Earth as a Planet, G. P. Kuiper editor, (U. of Chicago Press, Chicago 1954) pp. 57 - 137.

magnetized shell.

The earth's field shows spatial irregularities in regions of small extent on a geographical scale. They can be attributed to local concentrations of magnetic materials. If these anomalies are smoothed out, the resulting field can be analyzed in spherical harmonics. It is found that, to the accuracy that the analysis can be carried out from the observed data, the entire field is due to internal sources. The only parts of the field of external origin are rapidly varying components which are caused by solar or ionospheric disturbances.

A map of the average field shows no obvious correlation with the presence of land masses or other geological features. This feature suggests that the field must originate in the deep interior of the earth rather than in the crust. The earth's core is believed to be liquid, consisting principally of iron at a temperature of approximately 5000°K. Since the Curie point of iron is about 700°K, it would follow that ferromagnetic effects can be ruled out as a source of the terrestrial field. The only other known sources of magnetism are electric currents, and modern theories attempt to explain the earth's field on this basis.

1.2 Theories of Geomagnetism

If the terrestrial magnetic field is due to currents in the core, it is not clear at the outset that a source is needed to explain their presence. It was shown by Lamb[2], however, that currents in a

[2] H. Lamb, "On Electrical Motions in a Spherical Conductor," Trans. Roy. Soc. (London) 174, 519 (1883).

conducting sphere would decay with a lifetime of at most $\mu\sigma R^2/\pi^2$, where μ, σ are the permeability and conductivity of the sphere and R its radius. With values appropriate for the earth the decay time is approximately 15,000 years. Thus, in the absence of sources for the currents, the field would diminish by 1% every 150 years. This rate is not very large, but if the calculation is carried out into the distant past, it would require a field $10^{100,000}$ times larger than the present value, where the age of the earth has been taken as 3×10^9 years. It would thus appear more reasonable to assume that the energy lost by dissipation is compensated by internal electromotive forces. Of the various mechanisms which have been proposed to explain these sources the dynamo theory is of particular interest here.

The first suggestion that the earth's magnetic field might be caused by dynamo action was made by Larmor in 1919. In a short note[3] in which he discussed the possible origin of the magnetic fields of the sun, of sunspots, and of the earth, he said:

> "In the case of the sun, surface phenomena point to the existence of a residual internal circulation mainly in the meridional planes. Such internal motion induces an electric field acting on the moving matter: and if any conducting path around the solar axis happens to be open, an electric current will flow round it, which may in turn increase the inducing magnetic field. In this way it is possible for the internal cyclic motion to act after the manner of a self-exciting dynamo, and maintain a permanent magnetic field from insignificant beginnings, at the expense of some of the energy of the internal circulation. In any case,

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[3] J. Larmor, "How could a Rotating Body such as the Sun become a Magnet?" Rep. Brit. Assoc. Advanc. Sci. p. 159 (1919).

> in a celestial body residual circulation would be extremely permanent, as the large size would make effects of ordinary viscosity nearly negligible."

Larmor's suggestion was purely qualitative; he did not attempt to show that a dynamo such as he described was possible. In 1934, Cowling[4] formulated the dynamo problem in mathematical form and obtained a negative result which dealt a severe blow to the dynamo theories: he established that it is impossible for an axially symmetric field to be self maintained.

Cowling formulated the problem on the basis of the non-relativistic form of Maxwell's equations, in which the displacement current is neglected. For a homogeneous medium moving with velocity $\vec{v}$ the equations are:

$$\nabla \times \vec{B} = \mu \vec{J} \quad , \tag{1.1}$$

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t \quad , \tag{1.2}$$

$$\nabla \cdot \vec{B} = 0 \quad , \tag{1.3}$$

$$\vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B}) \quad . \tag{1.4}$$

These equations are easily combined into

$$\nabla \times \nabla \times B = \mu\sigma \left[-\frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{v} \times B) \right] \quad ,$$

or

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu\sigma} \nabla^2 \vec{B} \quad . \tag{1.5}$$

[4] T. G. Cowling, "The Magnetic Field of Sunspots," Monthly Notices Roy. Astron. Soc. 94, 39 (1934).

Equation (1.5) is known as the induction equation.

While a general statement of the problem should include the hydrodynamic equations for the motion of the medium, Cowling chose to regard the velocity field as given. In this case Eqs. (1.3) and (1.5) together with suitable subsidiary conditions are sufficient to determine the magnetic field that can exist with a particular flow pattern. If $\vec{v}$ is given as zero, then Eq. (1.5) becomes

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{\mu\sigma} \nabla^2 \vec{B} \quad , \tag{1.6}$$

which is the problem considered by Lamb in Ref. 2. In that case no steady fields exist; the solutions of Eq. (1.6) are of the form

$$\vec{B}(\vec{r},t) = \vec{B}(\vec{r})e^{-\alpha t} \quad , \tag{1.7}$$

where $\vec{B}(\vec{r})$ satisfies

$$\nabla^2 \vec{B} + \alpha\mu\sigma\vec{B} = 0 \quad . \tag{1.8}$$

It was noted earlier that the smallest value of α for a sphere of radius R is $\pi^2/\mu\sigma R^2$. This corresponds to the dipole mode. Higher multipoles decay more rapidly, and this may provide an explanation as to why planetary fields are essentially of dipole type: it is just the configuration that is dissipated most slowly and is thus the easiest to maintain.

Cowling chose to consider the case $\partial\vec{B}/\partial t = 0$, so that Eq. (1.5) becomes

$$\nabla \times \nabla \times \vec{B} = \mu\sigma \nabla \times (\vec{v} \times \vec{B}) \quad . \tag{1.9}$$

He was able to show that an axially symmetric magnetic field cannot be self-sustained. This result is known as Cowling's Theorem. Because of the importance of this result, and since it influenced all subsequent work, a demonstration of this theorem is given in the next section.

1.3 Cowling's Theorem

It will be shown in this section that steady axially symmetric dynamo action is impossible. The proof given will follow the arguments given by Cowling in Ref. 4. These are not entirely free of objections but they have the virtue of simplicity.

Using cylindrical coordinates (ρ, φ, z) with the z-axis along the axis of symmetry, and taking into account that derivatives with respect to φ must be zero, we write Eq. (1.3) as

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0 \quad . \tag{1.3a}$$

It follows that B_ρ and B_z can be derived from a scalar ψ:

$$B_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial z} \quad , \quad B_z = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} \quad . \tag{1.10}$$

$2\pi\psi(r, z)$ can be interpreted as the total magnetic flux crossing a circle of a radius r drawn on a plane at fixed z, since

$$\text{Flux} = \int \vec{B}\cdot d\vec{A} = 2\pi \int_0^\rho B_z \rho d\rho = 2\pi\left[\psi(\rho, z) - \psi(0, z)\right] \quad .$$

Now, the total flux crossing a closed surface is zero, and if the fields are supposed to be generated in a finite volume, the flux across any surface of infinite extent must be zero. Thus, $\psi(\infty, z)$ is constant and equal to $\psi(0, z)$. The constant can be chosen to be zero.

The function ψ thus defined is zero at the origin $\rho = 0$ and at $\rho = \infty$. It is also a continuous function, and therefore it is either zero everywhere, or it must have a maximum or a minimum at some point. At such a point

$$\frac{\partial\psi}{\partial\rho} = 0 \quad , \quad \frac{\partial\psi}{\partial z} = 0 \quad , \quad \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial\rho^2} \neq 0 \quad . \tag{1.11}$$

If the last condition is not satisfied the point is a saddle point rather than a maximum or minimum. It will now be shown that at all points where the first two conditions are satisfied the third is violated.

Since a steady dynamo is being considered, Eq. (1.2) reduces to

$$\nabla \times \vec{E} = 0$$

so that $\vec{E}$ can be written as the gradient of a scalar, and therefore $E_\varphi = 0$. The φ component of Eq. (1.4) is thus

$$J_\varphi = \sigma (v_z B_\rho - v_\rho B_z) \quad .$$

The quantities J_φ, B_ρ, and B_z can be expressed in terms of ψ; this results in

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\sigma}{\mu}\left[\frac{\partial\psi}{\partial z} v_z + \frac{\partial\psi}{\partial\rho} v_\rho\right] \quad , \tag{1.12}$$

so that

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

wherever both first derivatives are zero. It follows that $\psi = 0$ everywhere and thus $B_\rho = 0$, $B_z = 0$.

This proof does not state anything about B_φ. Later proofs of Cowling's Theorem have shown that such a field cannot be self-sustaining. In a paper published in 1957 Cowling[5] demonstrates that steady dynamo action is impossible for axially symmetric fields, two dimensional fields, and toroidal fields. These proofs are limited to the case $\nabla \cdot \vec{v} = 0$.

The implicit assumption is usually made that the flow field would have the same symmetry as the magnetic field, and for this reason Cowling's theorem is often stated in terms of the fluid velocity, i.e. that an axially symmetric flow cannot support dynamo action. Some proofs, however, are in terms of the velocity field: Backus[6] has shown that toroidal flows cannot maintain a steady dynamo.

1.4 The Possibility of Dynamo Action

After the enunciation of Cowling's theorem, work on the subject seems to have stopped, partly because of the advent of the war, and partly because of the implications of the theorem. Cowling himself seems to have become discouraged by his result, for in 1945 he

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[5] T. G. Cowling, "The Dynamo Maintenance of Steady Magnetic Fields", Quart. J. of Mech. App. Math. 10, 129 (1957).

[6] G. Backus, "A Class of Self-Sustaining Dissipative Spherical Dynamos", Ann. of Phys. 4, 372 (1958).

published a paper[7] in which he re-examined Lamb's result of Ref. 1, with a view to an alternative explanation of celestial magnetic fields. He concluded, though not enthusiastically, that the Sun's magnetic field might be the relic of a field that existed in the distant past, but that this would in any case not apply to the earth.

The investigation then fell into the hands of Elsasser, who was to influence greatly the subsequent work on dynamo theories. Although he had at first suggested[8] that the internal currents might be due to thermoelectric effects, Elsasser devoted a series of papers, the first of which appeared in 1946[9], to the dynamo effect.

His most significant contribution was to introduce a systematic method of attack for the spherical dynamo. He expanded both the magnetic field and the flow field in vector spherical harmonics. Because of the $\vec{v} \times \vec{B}$ term in the induction equation the harmonics are coupled so that what is finally obtained is an infinite set of coupled equations involving the radial functions which multiply the harmonics. He discussed the conditions under which dynamo action might result, and those under which it would not. In accordance with Cowling's theorem he found that "geometrically simple" configurations cannot give rise to steady dynamo action.

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[7] T. G. Cowling, "On the Sun's General Magnetic Field", Monthly Notices Roy. Astron. Soc. 105, 166 (1945).

[8] W. M. Elsasser, "On the Origin of the Earth's Magnetic Field", Phys. Rev. 55, 489 (1939).

[9] W. M. Elsasser, "Induction Effects in Terrestrial Magnetism", Phys. Rev. 69, 106 (1946).

Among other approaches tried by Elsasser, he developed a scheme for dealing with the steady dynamo which was used by later workers. He argued that if a given flow configuration $\vec{v}(\vec{r})$ is favorable for dynamo action, it does not necessarily produce a steady dynamo unless the magnitude of the velocity is just right. He therefore wrote the induction equation in the form

$$\nabla^2 \vec{B} + \lambda \nabla \times (\vec{v} \times \vec{B}) = 0 \quad , \tag{1.13}$$

where λ plays the role of an eigenvalue. Unfortunately, it cannot be shown that this equation must have a solution, or that, if a solution exists, the eigenvalue is real. It turns out, however, that for purposes of machine computation this form of the equation is convenient.

Elsasser also discussed the possible reasons for fluid motion in the interior of the earth. He emphasized the importance of Coriolis forces, and suggested that thermal convection coupled with these forces would probably tend to produce motion of the required type.

It is difficult to do justice to Elsasser's work in a short review. His papers, published over a period of ten years amount to several hundred pages. A review of his work was written by him in 1956[10].

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[10] W. M. Elsasser, "The Magnetic Field within the Earth"', Revs. Modern Phys. 28, 135 (1956).

Sir Edward Bullard devoted his attention to what he regarded to be the fundamental problem of dynamo theory: to prove that a solution to the dynamo equation exists. He first analyzed inductive effects in a rotating sphere[11]. He was able to obtain the remarkable result that the induction equation,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu\sigma} \nabla^2 \vec{B} \quad , \qquad (1.5)$$

remains unchanged when written with reference to a steadily rotating frame. This result is consistent with the theorem that toroidal velocity fields cannot support steady dynamo action, for rigid body rotation is a special case of toroidal motion, and the invariance of the equation implies that only the decaying fields described by Eq. (1.8) exist in this case. This result moreover implies that in the case of the earth it is only the relative motion between the core and mantle that can be responsible for dynamo action, and the rotation can be disregarded.

Bullard also studied the interaction of a rotating sphere with external fields. He found that while a dipole mode produces an internal field proportional to the angular speed, the higher spherical harmonics reach a maximum magnitude for the induced field and then decrease as the rotational speed increases further. This result was used ingeniously by Herzenberg in a paper to be described later.

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[11] E. C. Bullard, "Electromagnetic Induction in a Rotating Sphere", Proc. Roy. Soc. (London) 199, 413 (1949).

In an effort to prove the existence of a solution to Eq. (1.13), Bullard and Gellman[12] decided to attempt to exhibit the solution to one particular problem. They constructed a rather intricate velocity field which seemed likely to be complicated enough to lead to dynamo action, and at the same time might be an adequate model of the motion inside the earth if thermal convection is the dominating driving force. They then set up the problem for numerical calculation.

After 240 hours of machine time in one of the fastest electronic computers available at that time, they obtained a few iterated values for λ, for two slightly different velocity fields. For this numerical calculation the equations had to be cut off to give a finite system. A similar calculation was also carried out by Takeuchi and Shimazu[13], who in addition used a sixth order polynomial for the radial functions.

These results were regarded by Cowling to be convincing proof of the existence of a homogeneous dynamo[14], but Backus[6] severely criticized this stand. A passage from his paper is of interest:

> "Cowling writes that he is convinced of the existence of self-sustaining dynamos by the numerical computations carried out by Bullard and Gellman in an attempt to solve the eigenvalue problem

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[12] E. C. Bullard and H. Gellman, "Homogeneous Dynamos and Terrestrial Magnetism", Trans. Roy.Soc.(London) 247, 213 (1954).

[13] H. Takeuchi and Y. Shimazu, "On the Self-Exciting Process in Magneto-Hydrodynamics", J. Geophys. Res. 58, 497 (1953).

[14] T.G. Cowling, Magnetohydrodynamics (Interscience Publishers, New York, 1957) p. 87.

$\nabla \times \nabla \times \vec{B} = W(\vec{u} \times \vec{B})$ for the eigenvalue W and the eigenfunction B, given u. Such a solution would represent a steady dynamo. Bullard and Gellman . . . obtained successive approximations W_ℓ to the true eigenvalue W. Typical values for the W_ℓ they obtained for various velocity fields $\vec{u}$ are $W_1 = 22.06$, W_2 not computed, $W_3 = 67.4$. These sequences are supposed to converge to the true values of W; perhaps what Cowling finds convincing about them is that at least they are real. But as Chandresekhar has pointed out the steady increase of these approximate values of W as ℓ increases [ℓ indicates the harmonic at which the equations are truncated] and the approximation improves may indicate that in the exact solution and infinite value of W is required, or in other words that the particular velocities $\vec{u}$ chosen for the calculation cannot maintain a steady dynamo."

The question of the steady dynamo thus remained unsettled. In Ref. 6, Backus was able to show by very involved formal mathematical arguments that unsteady dynamo action is possible. His demonstration involved fluid motion with sudden starts and stops arranged in such a way as to satisfy various requirements in the steps of his development. While it proved the existence of such a dynamo, his discussion was not very tangible physically. Moreover, the real issue was that of steady dynamos, the existence of which had become doubtful because of Cowling's theorem.

This question was finally settled in 1958 by Herzenberg[15], who was able to devise a rather artificial, but easily visualized model capable of sustaining steady dynamo action. Herzenberg

[15] A. Herzenberg, "Homogeneous Dynamos", Trans. Roy. Soc. (London) 250, 543 (1958).

envisioned a system consisting of two small spheres A and B, imbedded in a larger sphere C. The three spheres are supposed to have the same conductivity and to be in perfect electrical contact with each other. Spheres A and B are assumed to be rotating at constant angular speeds about skew axes. By straightforward but lengthy mathematics, Herzenberg was able to show that for certain relative orientations of the two axes of rotation, sufficiently high angular speeds, and adequate separation of the small spheres, a steady field could be maintained. His procedure consisted of expanding the field in spherical harmonics referred to each of the three spheres. *In this way he showed that for example, the internal fields* of A and C produce an external dipole moment when referred to B. Because of B's rotation this would result in an induced dipole moment proportional to the rotational speed. This would in turn, when expressed in harmonics referred to A, provide an external dipole to generate the internal field of A, completing the regenerative cycle. Because of the saturation effect of the higher modes, mentioned in the discussion of Ref. 11, Herzenberg was able to demonstrate that in his case he was justified in keeping only the first few harmonics in these expansions.

The ideas involved in this model are fairly simple, but unfortunately the mathematical development is extremely complicated because of the inherent complexity of vector spherical harmonics which becomes almost prohibitive when the origin of the coordinate system is shifted or the orientation of the axes is changed. This work vindicated the suggestion of Larmor, who on defending his stand against an attack by Cowling wrote the following words, almost as if foreseeing the enormous efforts that were to follow:

> "This description [of the self sustaining dynamo] appears to be, on broad lines, so far as I can presently see, the inevitable conclusion, intricate enough, but one which mathematics could only obscure [16]."

1.5 The Question of Stability

Bullard[17] pointed out that it is not sufficient to establish the existence of a steady dynamo, but that it would also have to be demonstrated that it is stable. While this cannot be done with the induction equation alone since the magnetic field appears linearly throughout so that one field strength would satisfy the equation as well as any other, it would be of interest to find out whether the zero field configuration is stable. If the zero field is stable, it would mean that an initial perturbation would disappear, and dynamo action, while possible, would not occur spontaneously. If on the other hand the zero field situation is unstable, it would imply that the fluid motion would tend to amplify any stray field, presumably to a steady configuration, and that thus a sizable field could start from the "'insignificant beginnings'" as Larmor suggested.

So far the only work on stability has been on disc dynamos.

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[16] J. Larmor, "The Magnetic Field of Sunspots," Monthly Notices Roy. Astron. Soc. 94, 469 (1934).

[17] E. C. Bullard, "The Stability of a Homopolar Dynamo," Proc. Cambridge Phil. Soc. 51, 744 (1955).

Bullard studied the stability of a single disc, while Rikitake[18] and Lebovitz[19] have discussed the stability of a system of dynamos.

It would be desirable to carry out a stability analysis for a spherical homogeneous dynamo of the type described in the last section. The complexity of the analysis would be overwhelming, for the time dependent equations would have to be used, and the steady equations have proved to be almost impossible to handle. Nevertheless, it is to be hoped that an analysis of the stability of Herzenberg's dynamo will someday be carried out.

1.6 The Cause of Fluid Motion

The final aspect of importance in the dynamo theory of geomagnetism is to find out why the fluid in the core should be moving at all, and what motive power maintains this movement which would be resisted not only by friction, but also by the electromagnetic reaction which dynamo action would entail.

Bullard[20] and Elsasser[10], have suggested thermal convection, for it is known that there is a thermal gradient into the interior of the earth. This of course leaves the heat source to be explained.

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[18] T. Rikitake, "Oscillations of a System of Disk Dynamos," Proc. Cambridge Phil. Soc. 54, 89 (1958).

[19] N. R. Lebovitz, "The Equilibrium Stability of a System of Disk Dynamos," Proc. Cambridge Phil. Soc. 56, 154 (1960).

[20] E. C. Bullard, "The Magnetic Field within the Earth," Proc. Roy. Soc. (London) 197, 433 (1949).

Bullard maintains that radioactive decay inside the earth could be responsible for this. Urey[21] in advancing a rival theory argued that the amount of heat generated by radioactive decay would not be enough to make up the energy required to maintain the field. It may well turn out that a large part of the heat generated is by the currents that give rise to the magnetic field. Urey's theory is that heavier materials are being deposited continuously on the surface of the earth and they are slowly sinking into the center, thus producing a convection current.

In discussing the various alternatives, Bullard mentions a paper by Bondi and Lyttleton[22]. While the authors were concerned with the effects of the core on the dynamics of the rotation of the earth, and do not appear to have had geomagnetism in mind, Bullard considered precession as a possible source of the motion, but concluded that it would probably be too feeble a force.

The question remains still unsettled; it may well be that all of these causes contribute to some extent. It is the object of this work to analyze more fully the problem suggested by Bondi and Lyttleton, in the hope that the role played by the precession of the earth will become clearer.

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[21] H. C. Urey, "The Origin and Development of the Earth and other Terrestrial Planets," Geochim. Cosmochim. Acta, 1, 209 (1951)

[22] H. Bondi and R. A. Lyttleton, "On the Dynamical Theory of Rotation of the Earth. II. The Effect of Precession on the Motion of the Liquid Core," Proc. Cambridge Phil. Soc. 49, 498 (1953).

2. PRELIMINARY DISCUSSION OF THE PROBLEM

2.1 Description of Previous Work

Various problems involving rotating ellipsoidal masses of fluid have been studied for over a century. Poincaré and Kelvin[23] considered the perturbed motions of an incompressible fluid inside a cavity of small but non-zero eccentricity. In these studies the fluid was assumed to be inviscid. These authors concluded that the motion of such a fluid would be essentially that of a rigid body.

In more recent times, Bondi and Lyttleton devoted their attention to the part played by the liquid core of the earth in the dynamics of its motion. It is known that the earth has slight irregularities in its rotational motion[24]. There are three important systematic deviations from uniform rotation. First, the length of the day is increasing by approximately one millisecond per century because of tidal friction. Secondly, the gravitational action of the sun and moon on the equatorial bulge makes the axis of rotation precess about the normal to the ecliptic plane with a period of about 25,000 years, while maintaining an inclination of 23.4° to the normal. Finally, the earth has a "free body" precession with a period of approximately 400 days which is due to its oblateness. The precession axis makes an

[23] H. Lamb, Hydrodynamics (Dover Publications, New York, 1945), sixth edition, Chapter XII.

[24] Sir Harold Spencer Jones, "Dimensions and Rotation," in The Earth as a Planet, edited by G. P. Kuiper (The University of Chicago Press, Chicago 1954) pp 1 - 39.

angle of 10^{-6} radians with the rotation axis.

Bondi and Lyttleton reasoned that these motions would affect the flow of the liquid core so that, for example, the moments of inertia of the earth, which are calculated under the assumption of rigid body motion, would be in error. Their first paper[25] was devoted to the secular retardation of the core. They considered the perturbed motion of an incompressible fluid in a spherical cavity that is undergoing a slow change in its angular speed. They concluded that the motion of the core would follow that of the shell with a time lag of $(a^2/\omega\nu)^{\frac{1}{2}}$ where ν is the kinematic viscosity of the shell, ω its angular speed and <u>a</u> its internal radius. If one uses a value of ν of 100 m^2/sec, this retardation amounts to one year, but Bullard[20] has pointed out that the kinematic viscosity of the core material is probably much smaller than this. Bullard's estimate agrees with that given by Frenkel[26] and, since it is not very different from the kinematic viscosity of molten iron at ordinary pressures, 10^{-5} m^2/sec, this latter value will be used here. The time lag then becomes nearly 3000 years. It should be noted that in spite of this disparity the dimensionless parameter which measures the effect of the viscosity, $(\nu/\omega a^2)^{\frac{1}{2}}$, is small in either case, since it is 3×10^{-4} for $\nu = 100$ m^2/sec and 10^{-7} for $\nu = 10^{-5}$ m^2/sec.

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[25] H. Bondi and R. A. Lyttleton, "On the Dynamical Theory of the Rotation of the Earth I. The Secular Retardation of the Core," Proc. Cambridge Phil. Soc. 44, 345 (1948).

[26] J. Frenkel, Kinetic Theory of Liquids, (Dover Publications, New York 1955) p. 208.

In the problem of the core retardation, the change in angular velocity is parallel to the angular velocity itself so that an axially symmetric situation exists. This is no longer true for precession, since the angular acceleration is at right angles to the angular velocity, and the axial symmetry is thereby destroyed. From the point of view of dynamo theory only the second type of motion is a possible source of dynamo action.

The problem of flow in a precessing cavity was considered by Bondi and Lyttleton in Ref. 22. They were unable to come to a definite conclusion, but they provided a clear presentation of what they had been able to accomplish and of the difficulties they encountered. In their analysis they deliberately considered a spherical cavity rather than an ellipsoidal one to isolate the viscous effects from those arising from the shape of the container. Since in the spherical geometry the motion of the wall is at all times parallel to the wall, the only mechanism for transfer of momentum to the fluid is through viscous forces. In the case of an ideal fluid, the motion of the container would have no effect whatsoever on the enclosed fluid. Bondi and Lyttleton's approach to the problem was to consider the equations of motion in a frame which rotates in space at the precessional speed and to linearize the equations of motion about rigid body rotation, by taking the precessional speed as small. Their contention was that in this frame the motion would ultimately be steady, and it was this steady state that they were concerned with.

They assumed that the motion in the interior would be essentially unaffected by the viscosity so that the problem could be treated by a boundary layer approach. In attempting to solve the inviscid fluid equations for the interior, they found that, if it is required that the radial velocity vanish at the boundary, it is impossible for the velocity field to be represented by a function analytic throughout the interior of the sphere. Difficulties also arose in the boundary layer equations in that the boundary layer thickness became infinite on two circles at latitudes 30°N and 30°S. This complication arises in the present work also and it will be discussed more fully later.

Stewartson and Roberts[27] avoided these difficulties by posing the problem in a different way. They assumed that the container was initially rotating about a fixed axis, with the fluid rotating as a rigid body. The axis of rotation was set into a precessional motion about a given axis. They then asked what the subsequent motion of the fluid would be. They pointed out that, if the container is spherical, a frictionless fluid would continue to rotate about the original axis so that in the frame of reference considered by Bondi and Lyttleton the fluid velocity would be

$$\vec{u} = -\vec{\Omega} \times \vec{r} - (\vec{\Omega} \times \vec{\omega}) \times \vec{r} \sin \Omega t/\Omega \quad , \qquad (2.1)$$

where $\vec{\Omega}$ is the precessional velocity and $\vec{\omega}$ the rotational velocity. On linearizing this expression for small values of Ω, $\vec{u}$ becomes

$$\vec{u} = -\vec{\Omega} \times \vec{r} - (\vec{\Omega} \times \vec{\omega}) \times \vec{r}\, t \quad . \qquad (2.2)$$

[27]K. Stewartson and P. H. Roberts, "On the Motion of a Liquid Spheroidal Cavity of a Precessing Rigid Body," J. Fluid Mech. 17, 1 (1963).

This is an analytic expression that satisfies the linearized equations, but it is time dependent, and moreover it can only be valid for times which are small compared to the precessional period. It should be pointed out that this situation arises from the different approach to the problem used by Stewartson and Roberts, and that their result in no way precludes the existence of the steady state that was considered by Bondi and Lyttleton.

As far as the problem posed by Stewartson and Roberts is concerned, the complications just described made it necessary for them to consider an ellipsoidal container so that both viscous effects and the direct action of the walls enter into their analysis. Thus, their approach was to work the problem considered by Poincaré and Kelvin, but viscous effects were included. They were able to show that the steady solution obtained by Poincaré and Kelvin would be realized after a time of order a^2/ν, and that the flow would be as described by Poincaré and Kelvin except for a boundary layer flow and a negligible tertiary flow induced in the interior by the boundary layer.

The time required to establish steady flow seems disturbingly long since for the earth it could be as large as 10^{10} years, a time which indicates that the unsteady part of the flow could still exist in the core. In contrast with this result, it must be pointed out that the time required to communicate to the fluid a change in angular speed (without a change of axis) is $(a^2/\nu\omega)^{\frac{1}{2}}$ as obtained by Bondi and

Littleton[25], and later by Greenspan and Howard[28]. In the case of the earth this amounts to 3000 years. The appearance of two different time scales for phenomena of essentially similar nature is puzzling, and calls for more detailed study. In this work, however, only steady state conditions will be considered, so this issue will remain unresolved.

2.2 Outline of the Present Work

The work presented here follows the spirit of the investigation carried out by Bondi and Lyttleton. The container is taken to be a sphere, and steady state motion will be considered. In contrast with the case examined by these authors, however, the precession frequency will be allowed to have an arbitrary value. The angle between the precession and rotation axes will be taken to be small so that the equations may be linearized. This linearization appears necessary to make the analysis tractable. This is a departure from the original geophysical problem since that angle is 23.4° in the case of the earth. However, it will be found that there are certain advantages to this approach in that a distinction will become evident between slow and fast precession. In the latter type of flow it will be found that the problem can be solved for arbitrary values of the viscosity.

The case of highly viscous flow will be considered first, since it is the simplest to deal with. Then the problem with low viscosity

[28] H. P. Greenspan and L. N. Howard, "On a Time Dependent Motion of a Rotating Fluid," J. Fluid Mech. 17, 385 (1963).

will be considered and it will be seen that two types of flow arise. The flow for the rapidly precessing case will be treated for arbitrary viscosity. Then the flow for the case of slow precession, and low viscosity will be described. Finally, some conclusions will be drawn about the original problem posed by Bondi and Lyttleton in which the precession axis makes a large angle with the rotation axis.

3. DERIVATION OF THE EQUATIONS OF MOTION

3.1 Linearization of the Equations of Motion

The physical problem to be analyzed is the following: a spherical container, filled with an incompressible viscous fluid rotates with an angular velocity $\vec{\omega}_R$ about an axis which is itself rotating relative to an inertial frame with an angular velocity $\vec{\Omega}$. The vector $\vec{\omega}_R$ thus generates a cone whose axis is the vector $\vec{\Omega}$. The projection of $\vec{\omega}_R$ on $\vec{\Omega}$ is a fixed vector $\vec{\omega}_o$, while the part of $\vec{\omega}_R$ which is perpendicular to $\vec{\Omega}$, $\vec{\omega}_1$, has a sinusoidal time dependence. The half-angle of the cone generated by $\vec{\omega}_R$ is $\alpha = \tan^{-1}(\omega_1/\omega_o)$. This system will be described in the inertial frame in which the center of the sphere is at rest. Figure 1 illustrates the physical situation.

In this frame, the fluid velocity $\vec{q}$ satisfies the equation of continuity

$$\nabla \cdot \vec{q} = 0 \quad , \tag{3.1}$$

and the momentum equation

$$\frac{\partial \vec{q}}{\partial t} - \vec{q} \times (\nabla \times \vec{q}) - \nu \nabla^2 \vec{q} = -\nabla (p/\rho + \frac{1}{2} q^2) \quad , \tag{3.2}$$

where p is the pressure, ρ the density and ν the kinematic viscosity of the fluid. It should be noted that these equations are satisfied by a velocity field that represents rigid body rotation about a fixed axis, so that if $\omega_1/\omega_o << 1$, it would seem appropriate to linearize these equations about the flow field $\vec{\omega}_o \times r$ which is a solution to the equations of motion, but fails to satisfy the no-slip boundary conditions required by the problem under consideration.

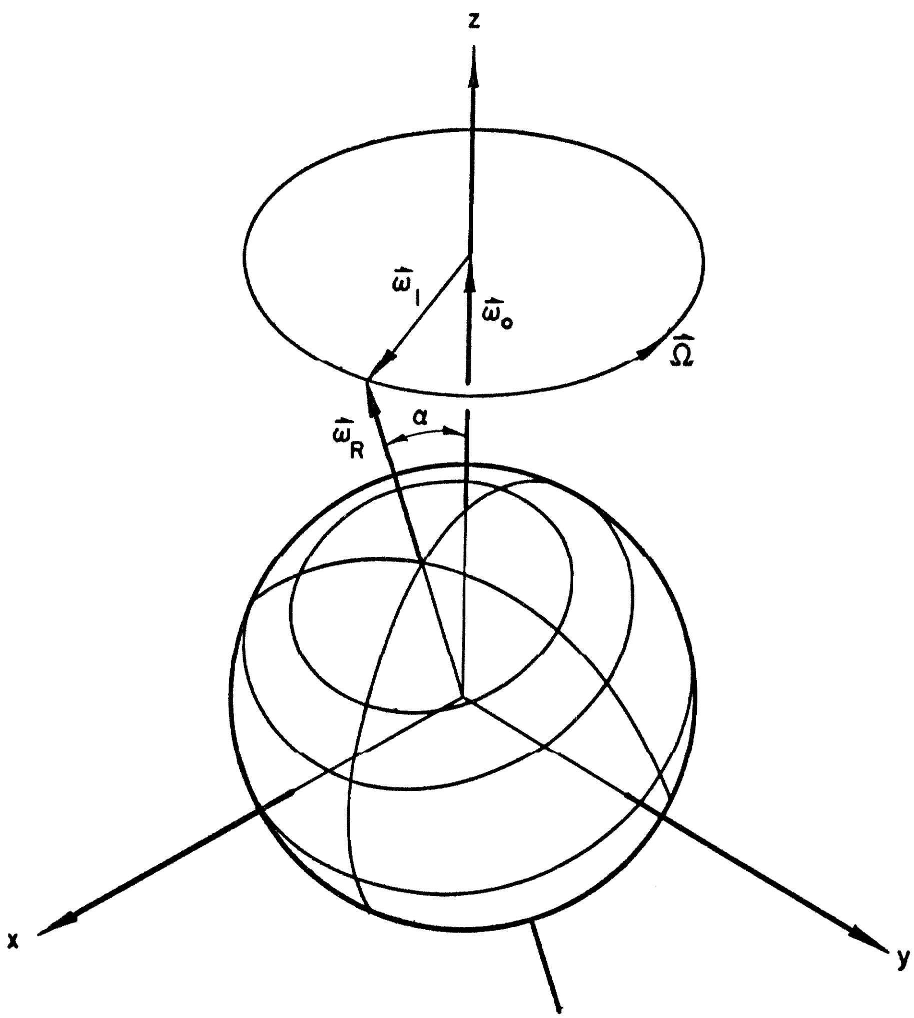

Figure 1. Geometry of the system.

To carry out the linearization, let

$$\vec{q} = \vec{\omega}_o \times \vec{r} + \vec{u} \quad . \tag{3.3}$$

Then Eq. (3.1) becomes

$$\nabla \cdot \vec{u} = 0 \quad , \tag{3.4}$$

where we have used the vector identity

$$\nabla \cdot (\vec{\omega}_o \times \vec{r}) = 0 \quad .$$

Similarly Eq. (3.2) becomes

$$\frac{\partial \vec{u}}{\partial t} - (\vec{\omega}_o \times \vec{r} + \vec{u}) \times \nabla \times (\vec{\omega}_o \times \vec{r} + \vec{u}) - \nu \nabla^2 (\vec{\omega}_o \times \vec{r} + \vec{u})$$

$$= - \nabla (p/\rho + \frac{1}{2} u^2 + \frac{1}{2} (\vec{\omega}_o \times \vec{r})^2 + \vec{u} \cdot (\vec{\omega}_o \times \vec{r})) \quad . \tag{3.5}$$

This can be simplified by making use of the following vector identities:

$$\nabla \times (\vec{\omega}_o \times \vec{r}) = 2\vec{\omega}_o \quad ,$$

$$\nabla^2 (\vec{\omega}_o \times \vec{r}) = 0 \quad ,$$

and

$$(\vec{\omega}_o \times \vec{r}) \times 2\vec{\omega}_o = 2(\vec{r}\, \omega_o^2 - \vec{\omega}_o (\vec{\omega}_o \cdot \vec{r})) \quad ,$$

$$= \nabla (\omega_o^2 r^2 - (\vec{\omega}_o \cdot \vec{r})^2) \quad ,$$

$$= \nabla (\vec{\omega}_o \times \vec{r})^2 \quad .$$

Equation (3.5) then becomes

$$\frac{\partial \vec{u}}{\partial t} - (\vec{\omega}_o \times \vec{r}) \times (\nabla \times \vec{u}) + 2\vec{\omega}_o \times \vec{u} - \vec{u} \times (\nabla \times \vec{u}) - \nu \nabla^2 \vec{u}$$

$$= - \nabla \left(p/\rho + \frac{1}{2} u^2 - \frac{1}{2} (\vec{\omega}_o \times \vec{r})^2 + \vec{u} \cdot (\vec{\omega}_o \times \vec{r}) \right) \quad . \qquad (3.6)$$

To linearize this equation we observe that

$$\vec{u} = 0 \quad , \qquad p = \frac{1}{2} \rho (\vec{\omega}_o \times \vec{r})^2 \quad ,$$

satisfies the equation. If ω_1 / ω_o is sufficiently small, $\vec{u}$ will also be a small quantity since it can be expected to be of the order of $\vec{\omega}_1 \times \vec{r}$. Similarly, the variable P defined by

$$\omega_o P = p/\rho - \frac{1}{2} (\vec{\omega}_o \times \vec{r})^2 \quad , \qquad (3.7)$$

which is zero when ω_1 is zero, can be taken to be proportional to ω_1 for small enough values of ω_1 / ω_o. Terms which are quadratic in ω_1 may thus be dropped from Eq. (3.6), and in this way the linearized momentum equation is obtained:

$$\frac{\partial \vec{u}}{\partial t} + \nabla [\vec{u} \cdot (\vec{\omega}_o \times \vec{r})] - (\vec{\omega}_o \times \vec{r}) \times (\nabla \times \vec{u}) + 2\vec{\omega}_o \times \vec{u} - \nu \nabla^2 \vec{u} = - \omega_o \nabla P \quad . \qquad (3.8)$$

The boundary condition for the problem is that the fluid adjacent to the container must be at rest with respect to the wall. In terms of $\vec{u}$, this requirement is

$$\vec{u} = \vec{\omega}_1 \times \vec{r} \qquad \text{at} \qquad r = a \quad , \qquad (3.9)$$

where $\underline{a}$ is the radius of the container.

3.2 The Equations of Motion in Spherical Coordinates

To refer these equations to a coordinate system, we choose an inertial frame with the z-axis along $\vec{\omega}_o$ (and $\vec{\Omega}$) and the origin at the center of the sphere. The x-axis and the time origin are chosen so that $\vec{\omega}_1$ takes on the form

$$\vec{\omega}_1 = \omega_1 \cos \Omega t \, \vec{e}_x + \omega_1 \sin \Omega t \, \vec{e}_y \quad . \tag{3.10}$$

In terms of the spherical coordinates (r, θ, φ), ω_1 then becomes

$$\vec{\omega}_1 = \omega_1 \sin\theta \cos(\varphi - \Omega t)\vec{e}_r + \omega_1 \cos\theta \cos(\varphi-\Omega t)\vec{e}_\theta - \omega_1 \sin(\varphi - \Omega t)\vec{e}_\varphi \quad . \tag{3.11}$$

while $\vec{\omega}_o$ is given by

$$\vec{\omega}_o = \omega_o \cos\theta \, \vec{e}_r - \omega_o \sin\theta \, \vec{e}_\theta \quad . \tag{3.12}$$

Similarly, we have

$$\vec{\omega}_o \times \vec{r} = \omega_o r \sin\theta \, \vec{e}_\varphi \quad ,$$

and

$$\vec{\omega}_o \times \vec{u} = -\omega_o \sin\theta u_\varphi \vec{e}_r - \omega_o \cos\theta u_\varphi \vec{e}_\theta + \omega_o(\cos\theta u_\theta + \sin\theta u_r)\vec{e}_\varphi \quad .$$

By substitution of these relations into Eq. (3.8) the following equations are obtained:

$$\frac{\partial u_r}{\partial t} + \frac{\partial}{\partial r}(u_\varphi \omega_o r \sin\theta) + \omega_o r \sin\theta \left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r u_\varphi)\right]$$

$$- 2\omega_o \sin\theta u_\varphi - \nu(\nabla^2 \vec{u})_r = -\omega_o \frac{\partial P}{\partial r} \quad ,$$

$$\frac{\partial u_\theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\varphi \omega_o r\sin\theta) - \omega_o r\sin\theta \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\varphi) + \frac{\partial u_\theta}{\partial\varphi}$$

$$-2\omega_o \cos\theta\, u_\varphi - \nu(\nabla^2\vec{u})_\theta = -\omega_o \frac{1}{r}\frac{\partial P}{\partial\theta} ,$$

and

$$\frac{\partial u_\varphi}{\partial t} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(u_\varphi \omega_o r\sin\theta) + 2\omega_o(\cos\theta\, u_\theta + \sin\theta\, u_r)$$

$$- \nu(\nabla^2\vec{u})_\varphi = -\frac{\omega_o}{r\sin\theta}\frac{\partial P}{\partial\varphi} .$$

There is a partial cancellation between the terms arising from $\nabla(\vec{u}\cdot(\vec{\omega}_o \times \vec{r}))$ and those from $(\vec{\omega}_o \times \vec{r}) \times (\nabla \times \vec{u})$, which is the reason for which the former of these terms was separated from the other terms involving the gradient of a scalar. With these cancellations, the equations simplify to

$$\frac{1}{\omega_o}\frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial\varphi} - 2\sin\theta u_\varphi - \frac{\nu}{\omega_o}(\nabla^2\vec{u})_r = -\frac{\partial P}{\partial r} , \qquad (3.13a)$$

$$\frac{1}{\omega_o}\frac{\partial u_\theta}{\partial t} + \frac{\partial u_\theta}{\partial\varphi} - 2\cos\theta\, u_\varphi - \frac{\nu}{\omega_o}(\nabla^2\vec{u})_\theta = -\frac{1}{r}\frac{\partial P}{\partial\theta} , \qquad (3.13b)$$

and

$$\frac{1}{\omega_o}\frac{\partial u_\varphi}{\partial t} + \frac{\partial u_\varphi}{\partial\varphi} + 2(\cos\theta\, u_\theta + \sin\theta\, u_r) - \frac{\nu}{\omega_o}(\nabla^2\vec{u})_\varphi = -\frac{1}{r\sin\theta}\frac{\partial P}{\partial\varphi} .$$

(3.13c)

The boundary conditions to be imposed at $r = a$, given by Eq. (3.9), when written out in component form, are

$$u_r = 0 , \qquad (3.14a)$$

$$u_\theta = -a\omega_1 \sin(\varphi - \Omega t) \quad , \qquad (3.14b)$$

and

$$u_\varphi = -a\omega_1 \cos(\varphi - \Omega t)\cos\theta \quad . \qquad (3.14c)$$

Since the boundary conditions are sinusoidal functions of the variable $\varphi - \Omega t$, we shall look for a steady state solution with the same dependence. It is convenient for this purpose to write $\vec{u}$ and P as the real part of complex terms as follows:

$$\vec{u} = \mathrm{Re}\{a\omega_1 \vec{v}(r,\theta)e^{i(\varphi-\Omega t)}\} \quad , \qquad (3.15)$$

and

$$P = \mathrm{Re}\{a^2\omega_1 Q(r,\theta)e^{i(\varphi-\Omega t)}\} \quad . \qquad (3.16)$$

The variables $\vec{v}$ and Q are mathematical constructs which are used to simplify the equations. We shall nevertheless call $\vec{v}$ the velocity and Q the pressure since those are the quantities that they determine. When there is a possibility of confusion the terms complex velocity and complex pressure will be used.

Another simplification is introduced into the equations of motion by writing them in terms of the dimensionless variables r' and t' defined by

$$r' = r/a \quad , \qquad (3.17)$$

and

$$t' = \omega_o t \quad . \qquad (3.18)$$

Since in most of the subsequent work these new variables will be used, the primes will be dropped with the understanding that it is the primed variables that are being used. Whenever it becomes necessary to refer to the unscaled variables a specific statement to that effect will be made.

With the above changes, the momentum equations become

$$-i(\omega-1)v_r - 2\sin\theta\, v_\varphi - \epsilon^2(\nabla^2\vec{v})_r = -\frac{\partial Q}{\partial r}\ , \tag{3.19a}$$

$$-i(\omega-1)v_\theta - 2\cos\theta\, v_\varphi - \epsilon^2(\nabla^2\vec{v})_\theta = -\frac{1}{r}\frac{\partial Q}{\partial \theta}\ , \tag{3.19b}$$

$$-i(\omega-1)v_\varphi + 2(\cos\theta\, v_\theta + \sin\theta\, v_r) - \epsilon^2(\nabla^2\vec{v})_\varphi = -\frac{iQ}{r\sin\theta} \tag{3.19c}$$

where $\omega = \Omega/\omega_o$ and $\epsilon^2 = \nu/a^2\omega_o$. The equation of continuity, in terms of $\vec{v}$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{iv_\varphi}{r\sin\theta} = 0\ . \tag{3.20}$$

These equations are subject to the boundary conditions

$$v_r = 0\ ,\quad v_\theta = i\ ,\quad v_\varphi = -\cos\theta\ , \tag{3.21}$$

at $r = 1$.

In Eq. (3.19) the components of the vector Laplacian ($\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times$) have not been written out explicitly to avoid undue complications. Because of the substitutions (3.15) and (3.16) the differential operator $\partial/\partial\varphi$ must be replaced by i in these terms.

3.3 The Equations of Motion in Cylindrical Coordinates

While the geometry of the problem is such that spherical coordinates are best to describe it, the equations of motion are most easily studied in the cylindrical coordinates (ρ, φ, z). With the same orientation for the axes as before, the vector $\vec{\omega}_1$ becomes

$$\vec{\omega}_1 = \omega_1 \cos(\varphi - \Omega t)\vec{e}_\rho - \omega_1 \sin(\varphi - \Omega t)\vec{e}_\varphi \quad , \tag{3.22}$$

while $\vec{\omega}_o$ is simply given by

$$\vec{\omega}_o = \omega_o \vec{e}_z \quad . \tag{3.23}$$

We then obtain

$$\vec{\omega}_o \times \vec{r} = \omega_o \rho \vec{e}_\varphi \quad ,$$

and

$$\vec{\omega}_o \times \vec{u} = - \omega_o u_\varphi \vec{e}_\rho + \omega_o u_\rho \vec{e}_\varphi \quad .$$

Upon substitution of these relations into Eq. (3.8) the following equations are obtained:

$$\frac{1}{\omega_o} \frac{\partial u_\rho}{\partial t} + \frac{\partial u_\rho}{\partial \varphi} - 2u_\varphi - \frac{\nu}{\omega_o} (\nabla^2 \vec{u})_\rho = - \frac{\partial P}{\partial \rho} \quad , \tag{3.24a}$$

$$\frac{1}{\omega_o} \frac{\partial u_\varphi}{\partial t} + \frac{\partial u_\varphi}{\partial \varphi} + 2u_\rho - \frac{\nu}{\omega_o} (\nabla^2 \vec{u})_\varphi = - \frac{1}{\rho} \frac{\partial P}{\partial \varphi} \quad , \tag{3.24b}$$

and

$$\frac{1}{\omega_o} \frac{\partial u_z}{\partial t} + \frac{\partial u_z}{\partial \varphi} - \frac{\nu}{\omega_o} (\nabla^2 \vec{u})_z = - \frac{\partial P}{\partial z} \quad . \tag{3.24c}$$

With the definitions of $\vec{v}$ and Q given in Eqs. (3.15) and (3.16), the components of the momentum equation in cylindrical coordinates reduce to

$$-i(\omega-1)v_\rho - 2v_\varphi - \epsilon^2 (\nabla^2 \vec{v})_\rho = - \frac{\partial Q}{\partial \rho} \quad , \tag{3.25a}$$

$$-i(\omega-1)v_\varphi + 2v_\rho - \epsilon^2 (\nabla^2 \vec{v})_\varphi = - \frac{iQ}{\rho} \quad , \tag{3.25b}$$

and

$$-i(\omega-1)v_z - \epsilon^2 \nabla^2 v_z = - \frac{\partial Q}{\partial z} \quad , \tag{3.25c}$$

where $\omega = \Omega/\omega_o$ and $\epsilon^2 = \nu/a^2\omega_o$ as before. In Eqs. (3.25a)and

(3.25b) the components of the vector Laplacian have not been written out in full. In Eq. (3.25c), however, the z-component of the vector Laplacian of $\vec{v}$ is the same as the scalar Laplacian of v_z so it has been written that way. Equation (3.25) has been written in terms of the dimensionless variables defined in Eqs. (3.17) and (3.18).

The corresponding form of the equation of continuity is

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v_\rho) + \frac{i v_\varphi}{\rho} + \frac{\partial v_z}{\partial z} = 0 \quad , \qquad (3.26)$$

and the boundary conditions are most simply expressed as follows

$$\left.\begin{array}{l} \rho v_\rho + z v_z = 0 \quad ; \quad z v_\rho - \rho v_z = i : \\ \qquad v_\varphi = - z \end{array}\right\} \text{at } \rho^2 + z^2 = 1 \quad . \qquad (3.27)$$

3.4 Reduction to a Single Equation

In Eqs. (3.4) and (3.8) the velocity components and the pressure are coupled. It is possible to combine these equations and to eliminate the velocity components. In this way a single scalar equation of sixth order is obtained for the pressure. Because it is of such a high order, the equation is not very useful. However, certain points about the problem can be made clearer by considering this equation rather than the coupled system.

To eliminate $\vec{u}$ from these equations we first take the divergence of Eq. (3.8):

$$\frac{\partial}{\partial t} (\nabla \cdot \vec{u}) + \nabla^2 [\vec{u} \cdot (\vec{\omega}_o \times \vec{r})] - (\nabla \times \vec{u}) \cdot 2\vec{\omega}_o + (\vec{\omega}_o \times \vec{r}) \cdot [\nabla \times (\nabla \times \vec{u})]$$

$$- 2\vec{\omega}_o \cdot (\nabla \times \vec{u}) - \nu \nabla^2 (\nabla \cdot \vec{u}) = - \omega_o \nabla^2 P \quad , \qquad (3.28)$$

where the following vector identities have been used:

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \quad ,$$

and

$$\nabla\cdot(\nabla^2\vec{A}) = \nabla^2(\nabla\cdot\vec{A}) \quad .$$

Now

$$\nabla\cdot\vec{u} = 0 \quad , \tag{3.4}$$

so that

$$\nabla^2(\vec{u}\cdot(\omega_o\times\vec{r})) - 4\vec{\omega}_o\cdot(\nabla\times\vec{u}) - (\vec{\omega}_o\times\vec{r})\cdot(\nabla^2\vec{u}) = -\omega_o\nabla^2 P \, . \tag{3.29}$$

A further reduction is obtained by making use of the following identities:

$$\nabla^2(\vec{r}\times\vec{A}) = \vec{r}\times(\nabla^2\vec{A}) + 2\nabla\times\vec{A} \quad ,$$

$$\nabla^2(\vec{c}\cdot\vec{A}) = \vec{c}\cdot\nabla^2\vec{A} \quad (\vec{c} \text{ constant}) \quad ,$$

and

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) \quad .$$

With the above

$$\begin{aligned}(\vec{\omega}_o\times\vec{r})\cdot\nabla^2\vec{u} &= \vec{\omega}_o\cdot(\vec{r}\times\nabla^2\vec{u}) \quad ,\\ &= \vec{\omega}_o\cdot[\nabla^2(\vec{r}\times\vec{u}) - 2\nabla\times\vec{u}] \quad ,\\ &= \nabla^2[\vec{\omega}_o\cdot(\vec{r}\times\vec{u})] - 2\vec{\omega}_o\cdot(\nabla\times\vec{u}) \quad ,\\ &= \nabla^2[\vec{u}\cdot(\vec{\omega}_o\times\vec{r})] - 2\vec{\omega}_o\cdot(\nabla\times\vec{u}) \quad .\end{aligned} \tag{3.30}$$

Equation (3.29) then simplifies to

$$2\vec{\omega}_o\cdot(\nabla\times\vec{u}) = \omega_o\nabla^2 P \quad . \tag{3.31}$$

In this equation the velocity appears only as a vorticity term. A second equation involving only the vorticity is obtained by taking the curl of Eq. (3.8); in fact, since only the component of $\nabla\times\vec{u}$ parallel to $\vec{\omega}_o$

appears in Eq. (3.31), we operate on Eq. (3.8) with $\vec{\omega}_o \cdot \nabla\times$. Thus, we obtain

$$\frac{\partial}{\partial t}\left[\vec{\omega}_o\cdot(\nabla\times\vec{u})\right] - \vec{\omega}_o\cdot\{\nabla\times[(\vec{\omega}_o\times\vec{r})\times(\nabla\times\vec{u})]\}$$

$$+2\vec{\omega}_o\cdot[\nabla\times(\vec{\omega}_o\times\vec{u})] - \nu\nabla^2[\vec{\omega}_o\cdot(\nabla\times\vec{u})] = 0 \quad . \qquad (3.32)$$

The triple vector product appearing in the second term of Eq. (3.32) can be expanded as follows:

$$(\vec{\omega}_o\times\vec{r})\times(\nabla\times\vec{u}) = \vec{r}(\vec{\omega}_o\cdot\nabla\times\vec{u}) - \vec{\omega}_o(\vec{r}\cdot\nabla\times\vec{u}) \quad ,$$

and making use of the identity

$$\nabla\times(g\vec{A}) = (\nabla g)\times\vec{A} + g\,\nabla\times\vec{A} \quad ,$$

we get

$$\vec{\omega}_o\cdot\{\nabla\times[(\vec{\omega}_o\times\vec{r})\times(\nabla\times\vec{u})]\} = (\vec{\omega}_o\times\vec{r})\cdot\nabla(\vec{\omega}_o\cdot\nabla\times\vec{u}) \quad .$$

The third term in Eq. (3.32) can be simplified by using the identities

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} \quad ,$$

and

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) + (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A}) \quad ,$$

from which we obtain

$$\vec{\omega}_o\cdot\nabla\times(\vec{\omega}_o\times\vec{u}) = -\vec{\omega}_o\cdot[(\vec{\omega}_o\cdot\nabla)\vec{u}] \quad ,$$

$$= -\vec{\omega}_o\cdot\nabla(\vec{\omega}_o\cdot\vec{u}) \quad .$$

With these changes, Eq. (3.32) becomes

$$\frac{\partial f}{\partial t} - (\vec{\omega}_o\times\vec{r})\cdot\nabla f - 2\vec{\omega}_o\cdot\nabla(\vec{\omega}_o\cdot\vec{u}) - \nu\nabla^2 f = 0 \quad , \qquad (3.33)$$

where

$$f = \vec{\omega}_o \cdot (\nabla \times \vec{u}) \quad .$$

A new scalar variable, $\vec{\omega}_o \cdot \vec{u}$, has appeared in this equation. If we take the coordinate system described before, with $\vec{\omega}_o = \omega_o \vec{e}_z$, the new variable is just $\omega_o u_z$, which appears coupled to P in Eq. (3. 24c). It can be easily verified that the operator $-(\vec{\omega}_o \times \vec{r}) \cdot \nabla$ when written in cylindrical coordinates is simply $\omega_o \partial/\partial\varphi$ so that we can rewrite Eqs. (3. 31), (3. 33) and (3. 24c) in the following form:

$$2f = \omega_o \nabla^2 P \quad , \tag{3. 34}$$

$$\frac{\partial f}{\partial t} + \omega_o \frac{\partial f}{\partial \varphi} - \nu \nabla^2 f = 2\omega_o^2 \frac{\partial u_z}{\partial z} \quad , \tag{3. 35}$$

and

$$\frac{\partial u_z}{\partial z} + \omega_o \frac{\partial u_z}{\partial \varphi} - \nu \nabla^2 u_z = -\omega_o \frac{\partial P}{\partial z} \quad . \tag{3. 24c}$$

The variables f and u_z are easily eliminated by repeated differentiations and the final form of the equation for P is

$$\left[\frac{1}{\omega_o} \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} - \frac{\nu}{\omega_o} \nabla^2 \right]^2 \nabla^2 P + 4 \frac{\partial^2 P}{\partial z^2} = 0 \quad . \tag{3. 36}$$

This equation can also be written in terms of the variable Q defined in Eq. (3.16). The equation with dimensionless length and time scales is

$$(i(\omega-1) + \epsilon^2 \nabla^2)^2 \nabla^2 Q + 4 \frac{\partial^2 Q}{\partial z^2} = 0 \quad . \tag{3. 37}$$

3.5 Discussion of the Equations

The equations of motion for the fluid, as given in sections 3.2 and 3.3 constitute a sixth order system. From Eq. (3.36) it is evident that the system is elliptic, and thus the three boundary conditions given on a closed boundary make this a well-defined problem. The angular dependence and the time dependence can be factored out because the equations have been linearized, and thus the problem is essentially a two dimensional one. The solution of the problem presents two kinds of difficulties. First, the geometry is such that the equations are separable in cylindrical coordinates, as is readily seen from Eq. (3.36), but they are not separable in spherical coordinates. On the other hand, the boundaries are not appropriate for a treatment in cylindrical coordinates. Thus, a solution by the methods of separation of variables is impossible in this case, except for the approximate expansion which is the subject of the next chapter. This expansion is valid for high values of the viscosity and will thus be referred to as the creeping flow approximation.

The second difficulty in this problem is that the equations are of such a high order. The alternate method of approximation that can be employed is valid for small values of the viscosity, since the highest derivatives appear in the equations multiplied by powers of the viscosity. Thus, for sufficiently small values of the viscosity the order of the equations can be reduced. This is the object of the boundary layer solution studied in Chapters 5 and 6.

4. CREEPING FLOW SOLUTIONS

4.1 Expansion in Inverse Powers of the Viscosity

The equations of motion, in terms of the complex velocity $\vec{v}$ and the complex pressure Q, can be written in the following way:

$$\epsilon^2 \nabla^2 \vec{v} + i(\omega - 1)\vec{v} = A\vec{v} + \nabla Q \quad , \tag{4.1}$$

where A is an operator which in spherical coordinates is represented by the matrix

$$\begin{pmatrix} 0 & 0 & -2\sin\theta \\ 0 & 0 & -2\cos\theta \\ 2\sin\theta & 2\cos\theta & 0 \end{pmatrix} \quad , \tag{4.2}$$

as can be determined by referring to Eq. (3.19). In addition, the velocity satisfies the continuity equation

$$\nabla \cdot \vec{v} = 0 \quad , \tag{4.3}$$

and the boundary conditions

$$\left. \begin{aligned} v_r &= 0 \\ v_\theta &= i \\ v_\varphi &= -\cos\theta \end{aligned} \right\} \quad \text{at} \quad r = 1 \quad . \tag{3.21}$$

It was shown by Lamb[29] that a solenoidal vector field such as $\vec{v}$ can be split in a unique way into the sum of two partial fields, one derived from a scalar potential T and the other from a scalar potential

[29] H. Lamb, "On the Oscillations of a Viscous Spheroid," Proc. London Math. Soc. 13, 51 (1881).

S in the following manner

$$\vec{v} = \nabla \times (\vec{r}\, T) + \nabla \times \nabla \times (\vec{r}\, S) \quad . \tag{4.4}$$

Elsasser has called the partial field derived from T a toroidal field and the one derived from S a poloidal field. Equation (4.4) then says that a vector whose divergence is zero can be split into a toroidal and a poloidal part. It has been demonstrated by Backus[6] that this representation is complete in the interior of a sphere provided the flux of the vector field is zero across the surface of the sphere.

Since the divergence operator commutes with the vector Laplacian, it follows that if $\vec{v}$ is solenoidal, then $\nabla^2\vec{v}$ is also solenoidal, and therefore must have a toroidal and a poloidal part. It can be shown that if $\vec{v}$ is given by Eq. (4.4), then

$$\nabla^2 \vec{v} = \nabla \times (\vec{r}\, \nabla^2 T) + \nabla \times \nabla \times (\vec{r}\, \nabla^2 S) \quad , \tag{4.5}$$

so that the toroidal and poloidal parts of $\nabla^2\vec{v}$ are derived from scalar potentials which are obtained by taking the Laplacian of the corresponding scalar potentials from which $\vec{v}$ is derived. This fact was discovered by Lamb, who used it in an ingenious way to uncouple the vector Helmholtz equation into two scalar Helmholtz equations. We shall use this procedure to obtain a solution to our problem by successive approximations.

In Eq. (4.1) the left hand side contains only terms proportional to $\vec{v}$ and $\nabla^2 \vec{v}$, so that the toroidal and poloidal scalar potentials are uncoupled. The presence of the term ∇Q in the right hand side is not serious, since this term could be removed by taking the curl of

Eq. (4.1), in which case the left hand side could still be separated into two uncoupled parts. The term $A\vec{v}$, however, couples the velocity components in such a way that the two scalar potentials are intermixed. This suggests that a fruitful approach to the problem may be to first neglect this term to obtain an approximate solution for $\vec{v}$, and to then use this (known) value of $\vec{v}$ in the term $A\vec{v}$ to obtain a second approximation. This procedure is equivalent to an expansion in inverse powers of ϵ^2 and this will be pursued here. An alternate choice will be examined in section 4.4.

To proceed with this approach, we first expand all the variables in inverse powers of ϵ^2, so that

$$\vec{v} = \vec{v}_o + \epsilon^{-2}\vec{v}_1 + \epsilon^{-4}\vec{v}_2 + \ldots \quad , \tag{4.6}$$

with similar expansions for T and S. It will be assumed that each of the $\vec{v}_n$ satisfies the divergence condition individually, and that $\vec{v}_o$ satisfies the boundary conditions (3.21), while the other $\vec{v}_n$ are subject to homogeneous boundary conditions. Since it is not obvious a priori that $\vec{v}_o$ can satisfy $\nabla^2\vec{v}_o = 0$ and the boundary conditions, although this is in fact the case, the expansion for Q must take the form

$$Q = \epsilon^2 Q_{-1} + Q_o + \epsilon^{-2}Q_1 + \ldots \quad . \tag{4.7}$$

If these expansions for $\vec{v}$ and Q are introduced into Eq. (4.1) and the coefficient of each power of ϵ^2 is set equal to zero separately, the following sequence of equations results:

$$\nabla^2 \vec{v}_o = \nabla Q_{-1} \quad , \tag{4.8}$$

$$\nabla^2 \vec{v}_n = i(\omega-1)\vec{v}_{n-1} + A\vec{v}_{n-1} + \nabla Q_{n-1}, \quad (n = 1, 2, \ldots) \quad . \tag{4.9}$$

Equation (4.9) gives $\vec{v}_n$ if both $\vec{v}_{n-1}$ and Q_{n-1} are known, but we must also be able to determine Q_n to have a complete system of equations so as to proceed to higher iterations. The missing relation is obtained by taking the divergence of Eq. (4.9), which gives

$$\nabla^2 Q_{n-1} = 2(\nabla \times \vec{v}_{n-1})\cdot\vec{e}_z \quad , \tag{4.10}$$

a result which could have been obtained directly from Eq. (3.33). The problem then reduces to successive solutions of Poisson's equation. This description is, however, deceptively simple, for boundary conditions are not immediately available to determine the unique solution at each step. The uncertainties in the solution must be carried until the boundary condition can be applied. Thus, for example, if $\vec{v}_{n-1}$ has been found, Eq. (4.10) determines Q_{n-1} within an arbitrary additive harmonic function. The boundary conditions, however, do not say anything about Q_{n-1}, so that the harmonic function must be carried along until $\vec{v}_n$ is determined, at which stage the homogeneous boundary condition can be applied. The trouble stems from the fact that, strictly speaking, we are not dealing with iterations with Poisson's equation but with the inhomogeneous biharmonic equation. We shall see that, in terms of the scalar potentials, T_n satisfies Poisson's equation while S_n satisfies an inhomogeneous biharmonic equation.

To proceed to the details of this solution it is necessary to express the components of $\vec{v}$ and $\nabla^2\vec{v}$ in spherical coordinates in terms of T and S. The necessary relations are given by

Chandrasekhar[30] among others. These relations for $\vec{v}$ are

$$\left.\begin{aligned} v_r &= \frac{1}{r} L^2 S \quad , \\ v_\theta &= \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} (rS) + \frac{iT}{\sin\theta} \quad , \\ v_\varphi &= \frac{i}{r\sin\theta} \frac{\partial}{\partial r} (rS) - \frac{\partial T}{\partial \theta} \quad , \end{aligned}\right\} \tag{4.11}$$

while the components of $\nabla^2 \vec{v}$ are obtained by replacing T and S by $\nabla^2 T$ and $\nabla^2 S$ respectively. The operator L^2 which appears in the first of these relations is the angular part of the Laplacian operator, sometimes called the Legendrian:

$$L^2 = - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \quad . \tag{4.12}$$

It should be remarked that these special forms which apply only to the case in which the φ dependence is given by $e^{i\varphi}$. The boundary conditions, expressed in terms of T and S, become

$$\left.\begin{aligned} T_o &= \sin\theta \quad , \quad T_n = 0 \quad (n = 0) \quad , \\ S_n &= 0 \quad , \quad \frac{\partial S_n}{\partial r} = 0 \quad , \end{aligned}\right\} \text{ at } r = 1 \quad . \tag{4.13}$$

Thus T satisfies only one condition while S has to satisfy two. The equations of motion, in terms of T and S are rather complicated, and moreover are not in the most convenient form for solution of the problem. Instead of writing them in component form

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[30] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability (Oxford University Press, London, 1961) pp. 622-626.

we shall go through the steps required to carry out one iteration, assuming $\vec{v}_{n-1}$ is known, or equivalently, T_{n-1} and S_{n-1}.

The first step is to find Q_{n-1} by using Eq. (4.10). This equation can be written in terms of T_{n-1} and S_{n-1} as follows:

$$\nabla^2 Q_{n-1} = \frac{2}{r} \cos\theta L^2 T_{n-1} - \sin\theta \frac{\partial^2}{\partial r \partial\theta} (rT_{n-1}) + 2i\nabla^2 S_{n-1} \quad . \tag{4.14}$$

Let $\tilde{Q}_{n-1}$ be any particular integral of this equation. Then Q_{n-1} can differ from $\tilde{Q}_{n-1}$ only by a harmonic function, which we choose to write as $\partial(r\chi_{n-1})/\partial r$, so that

$$Q_{n-1} = \tilde{Q}_{n-1} + \frac{\partial}{\partial r} (r\chi_{n-1}) \quad . \tag{4.15}$$

The reason for writing the harmonic function in this way is that we shall want to separate its gradient (which is a solenoidal vector) into a toroidal and a poloidal vector, and this is most easily done by writing the harmonic function in this way, as was done by Lamb[29]. To show this we first make use of the vector identity

$$\nabla^2(\vec{r} \cdot \nabla\psi) = \vec{r} \cdot \nabla(\nabla^2\psi) + 2\nabla^2\psi \quad ,$$

from which it follows that if ψ is harmonic so is $\partial(r\psi)/\partial r$. A second vector identity

$$\nabla\times\nabla\times(\vec{r}\psi) = \nabla(\psi + \vec{r}\cdot\nabla\psi) - \vec{r}\nabla^2\psi \quad ,$$

establishes that, if ψ is harmonic, then $\nabla\partial(r\psi)/\partial r$ is a purely poloidal vector derived from the scalar potential ψ.

With $\tilde{Q}_{n-1}$ a known function, Eq. (4.9) can be written as

$$\nabla^2\vec{v}_n = -i(\omega-1)\vec{v}_{n-1} + \nabla\times\nabla\times(\vec{r}\chi_{n-1}) + A\vec{v}_{n-1} + \nabla\widetilde{Q}_{n-1} \quad , \qquad (4.16)$$

where χ_{n-1} is unknown, but is such that

$$\nabla^2\chi_{n-1} = 0 \quad . \qquad (4.17)$$

Since all the other terms in Eq. (4.16) are solenoidal, the vector $A\vec{v}_{n-1} + \nabla Q_{n-1}$ must also be solenoidal and can thus be split into a toroidal part t_{n-1} and a poloidal part, s_{n-1}. To effect this separation we use Eq. (4.11). The poloidal part, s_{n-1}, is obtained from the radial component of the vector:

$$L^2 s_{n-1} = \vec{r}\cdot[A\vec{v}_{n-1} + \nabla\widetilde{Q}_{n-1}] \quad . \qquad (4.18)$$

The operator L^2 is easily inverted if spherical harmonics are used. Once the poloidal part is known, the toroidal part is obtained from the θ-component of the vector:

$$t_{n-1} = -i \sin\theta[\vec{e}_\theta\cdot(A\vec{v}_{n-1} + \nabla\widetilde{Q}_{n-1}) - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(rs_{n-1})] \quad . \qquad (4.19)$$

The differential equations for T_n and S_n can then be obtained directly from Eq. (4.9). They are

$$\nabla^2 T_n = -i(\omega-1)T_{n-1} + t_{n-1} \quad , \qquad (4.20)$$

and

$$\nabla^2 S_n = -i(\omega-1)S_{n-1} + s_{n-1} + \chi_{n-1} \quad . \qquad (4.21)$$

In Eq. (4.21) the harmonic function χ_{n-1} could be left undetermined until S_n has been evaluated. Then the two boundary conditions imposed on S_n would determine both S_n and

χ_{n-1}. It is possible, however, to determine χ_{n-1} first. This is accomplished by multiplying Eq. (4.21) by χ_{n-1} and integrating over the volume of the sphere. The result is

$$\int \chi_{n-1} \nabla^2 S_n dV = \int [-i(\omega-1)S_{n-1} + s_{n-1} + \chi_{n-1}]\chi_{n-1} dV \quad .$$

Since $\nabla^2 \chi_{n-1} = 0$, a term $S_n \nabla^2 \chi_{n-1}$ can be subtracted from the left hand integrand without changing its value. Thus,

$$\int \chi_{n-1} \nabla^2 S_n dV = \int (\chi_{n-1} \nabla^2 S_n - S_n \nabla^2 \chi_n) dV = \int (\chi_n \nabla S_n - S_n \nabla \chi_n) \cdot d\vec{S}$$

by Green's theorem. Applying the boundary conditions (4.13) to S_n, we find that the surface integral is zero, so that

$$\int \chi_{n-1}^2 dV = \int [i(\omega-1)S_{n-1} - s_{n-1}] \chi_{n-1} dV \quad . \tag{4.22}$$

Now suppose that χ_{n-1} is expanded in spherical harmonics,

$$\chi_{n-1} = \sum a_\ell \psi_\ell \quad , \tag{4.23}$$

where

$$\psi_\ell = r^\ell P_\ell^1 (\cos\theta) \quad .$$

Since the ψ_ℓ are orthogonal, substitution into Eq. (4.22)yields

$$a_\ell^2 \int \psi_\ell^2 dV = a_\ell \int [i(\omega-1)S_{n-1} - s_{n-1}]\psi_\ell dV \quad ,$$

so that

$$a_\ell = \frac{\int [i(\omega-1)S_{n-1} - s_{n-1}]\psi_\ell \, dV}{\int \psi_\ell^2 \, dV} \quad . \tag{4.24}$$

Thus χ_{n-1} may be determined through Eqs. (4.23) and (4.24) so that the entire right hand side is known in Eq. (4.21). S_n is then determined by applying either of the boundary conditions (4.13). In this way one iteration is complete, and, at least in principle, as many iterations as are desired can be carried through.

The zeroth order solution that starts the entire process must now be determined. The pertinent equations are:

$$\nabla^2 Q_{-1} = 0 \quad ,$$

$$\nabla^2 S_o = \chi_{-1} \quad ,$$

and

$$\nabla^2 T_o = 0 \quad .$$

It is clear that $\tilde{Q}_{-1} = 0$ is a particular integral of the first of these. From (4.24) we obtain $\chi_{-1} = 0$ and thus $S_o = 0$. Finally, since T_o must have a $\sin\theta$ angular dependence to satisfy the boundary condition, we find

$$T_o = r \sin\theta \quad .$$

In Appendix A the details of the first iteration are given. The results of the second iteration are also quoted. The components of the velocity field thus obtained are

$$v_r = -\frac{1}{\epsilon^4}\frac{i\omega r}{420}(r^2-1)^2 \sin\theta\cos\theta \quad , \tag{4.25a}$$

$$v_\theta = i\,r + \frac{1}{\epsilon^2}\frac{\omega r}{10}(r^2-1)$$

$$-\frac{1}{\epsilon^4}\left[\frac{i\omega r}{2520}(7r^2-3)\cos 2\theta + \frac{i\omega^2 r}{1400}(5r^2-9)\right](r^2-1) \quad , \tag{4.25b}$$

and

$$v_\varphi = -r\cos\theta + \frac{1}{\epsilon^2}\frac{i\omega r}{10}(r^2-1)\cos\theta$$

$$+\frac{1}{\epsilon^4}\left[\frac{\omega r}{2520}(7r^2-3) + \frac{\omega^2 r}{1400}(5r^2-9)\right](r^2-1)\cos\theta \quad . \tag{4.25c}$$

The expression for Q obtained from two iterations goes only as far as terms proportional to ϵ^{-2}. To this accuracy, Q is

$$Q = -r^2 \sin\theta\cos\theta + \frac{1}{\epsilon^2}\frac{i\omega r^2}{105}(3r^2-7)\sin\theta\cos\theta \quad . \tag{4.26}$$

4.2 Discussion

The expressions for the velocity given in Eq. (4.25) consist, first of all, of the zeroth order approximation which corresponds to rigid body motion. This is the exact solution of the equations of motion for the case of either infinite viscosity ($\epsilon = \infty$), or, alternatively, for infinitely slow precession ($\omega = 0$). The first correction due to the finite value of ϵ affects only the θ and φ components of the velocity. It represents a toroidal flow taking place in the body of the fluid. It is noteworthy that the first poloidal component does not appear until the second correction to rigid body flow.

The second correction also shows the first appearance of terms which are not linear in ω. These are due to the term $(\omega-1)\vec{v}$ in the equations of motion. Evidently, higher powers will appear in subsequent terms, and, in fact, it can be seen from Eq. (4.9) that ω will appear in the n-th term, $\vec{v}_n$, raised to powers up to the n-th. Thus, although the expansion was formulated as one in inverse powers of the viscosity, it also turns out to have the property that a given power of ω does not appear until the term of the corresponding order. This suggests that an expansion based on the scheme

$$\nabla^2 \vec{v}_n + \frac{i(\omega-1)}{\epsilon^2} \vec{v}_n = A\vec{v}_{n-1} + \nabla Q_{n-1} \tag{4.27}$$

might be worth pursuing. This is an expansion in inverse powers of the viscosity, but with $(\omega-1)/\epsilon^2$, rather than ω, as the second parameter. This approach is explored in Section 4.4. It has the disadvantage that the dependence on r is no longer expressed in terms of polynomials. Nevertheless, it leads to an interesting connection with the boundary layer solution to be discussed in the next chapter. Before proceeding to that stage, however, we shall study further some aspects of the solution just obtained.

4.3 Dynamical Consequences

The results contained in Eqs. (4.25) and (4.26) cannot be conveniently represented in a figure, not only because of the three dimensional character of the flow, but also because there are two parameters whose values would have to be assigned. An idea of the behavior of the fluid as a whole is obtained from the response of the liquid sphere to the torques that maintain the precession of the shell.

Thus, for example, the total angular momentum of the fluid will differ from that of a similar homogeneous rigid body because the fluid is undergoing internal motions. Furthermore, these internal motions result in viscous dissipation of energy so that, if the motion is to remain steady, the external torques must do work on the fluid. For this reason, the torque acting on the fluid sphere will not be at right angles to the angular velocity of the shell, as would be the case for a rigid sphere undergoing a similar precessional motion. In this section, we propose to study the dynamical properties of the fluid sphere as a whole.

The total angular momentum of the fluid sphere about the origin is

$$\vec{L} = \rho \int \vec{r} \times \vec{q} \, dV \quad , \tag{4.28}$$

where $\vec{q}$ is the actual fluid velocity in the inertial frame. In Eq. (3. 3), $\vec{q}$ was split into a part which arises from the uniform rotation about a fixed axis, $\vec{\omega}_o \times \vec{r}$, and a part $\vec{u}$ due to the precessing component of the total angular velocity of the shell. The contribution to the angular momentum due to the first part is simply

$$\vec{L}_o = \frac{8}{15} \pi \rho a^5 \vec{\omega}_o \quad , \tag{4.29}$$

since $8\pi\rho a^5/15$ is the moment of inertia of a homogeneous rigid sphere about any axis passing through its center. The remaining part of the angular momentum we shall call $\vec{L}_p$, where

$$\vec{L}_p = \rho \int \vec{r} \times \vec{u} \, dV \quad . \tag{4.30}$$

In these equations the variables that appear are the actual physical variables and the lengths are unscaled. In the preceding sections we have been dealing with dimensionless lengths and the complex velocity $\vec{v}$ which was introduced by the equation

$$\vec{u} = \mathrm{Re}\{a\omega_1 \vec{v} e^{i(\varphi - \Omega t)}\} \quad . \tag{3.15}$$

The angular momentum can similarly be written in terms of a complex angular momentum $\vec{\ell}_p$ defined by the relation

$$\vec{L}_p = \mathrm{Re}\{\rho a^5 \omega_1 \vec{\ell}_p e^{-i\Omega t}\} \quad , \tag{4.31}$$

where

$$\vec{\ell}_p = \int \vec{r} \times \vec{v} \, dV \quad . \tag{4.32}$$

The lengths are now expressed in the dimensionless units defined in Eq. (3.17) in terms of which the sphere has unit radius.

The complex angular momentum $\vec{\ell}_p$ can now be expanded in powers of ϵ^{-2} as follows

$$\vec{\ell}_p = \vec{\ell}_o + \epsilon^{-2}\vec{\ell}_1 + \epsilon^{-4}\vec{\ell}_2 + \ldots \quad , \tag{4.33}$$

where

$$\begin{aligned} \vec{\ell}_n &= \int \vec{r} \times \vec{v}_n \, dV \quad , \\ &= \int [-r v_{n\varphi} \vec{e}_\theta + r v_{n\theta} \vec{e}_\varphi] dV \quad . \end{aligned} \tag{4.34}$$

To find the first few terms in this expansion we shall require the value of certain integrals which involve the unit vectors $\vec{e}_\theta$ and $\vec{e}_\varphi$. These integrals are

$$\vec{a}_1 = \int \vec{e}_\theta \cos\theta\, e^{i\varphi} d\Omega = \frac{2\pi}{e}\ \vec{e}_+ \quad ,$$

$$\vec{a}_2 = \int \vec{e}_\varphi e^{i\varphi} d\Omega = -2\pi i \vec{e}_+ \quad ,$$

and

$$\vec{a}_3 = \int \vec{e}_\varphi \cos 2\theta\, e^{i\varphi} d\Omega = \frac{2\pi i}{3}\ \vec{e}_+ \quad ,$$

where

$$\vec{e}_+ = \vec{e}_x + i\,\vec{e}_y \quad .$$

To show how these integrals were obtained we evaluate $\vec{a}_1$. By expanding $\vec{e}_\theta$ into its rectangular components, $\vec{a}_1$ becomes

$$\vec{a}_1 = \int_0^\pi \cos\theta \sin\theta\, d\theta \int_0^{2\pi} (\vec{e}_x \cos\theta\cos\varphi + \vec{e}_y \cos\theta\sin\varphi - \vec{e}_z \sin\theta) e^{i\varphi} d\varphi \quad ,$$

$$= \int_0^\pi \cos\theta \sin\theta (\pi \vec{e}_x \cos\theta + i\pi \vec{e}_y \sin\theta) d\theta \quad ,$$

$$= \frac{2\pi}{3}\ (\vec{e}_x + i\,\vec{e}_y) \quad .$$

The other two integrals are found in a similar manner.

With the help of these integrals, we shall now evaluate $\vec{\ell}_0$, $\vec{\ell}_1$, and $\vec{\ell}_2$. First, $\vec{\ell}_0$ is given by

$$\vec{\ell}_0 = \int (-r v_{0\varphi} \vec{e}_\theta + r v_{0\theta} \vec{e}_\varphi)\, dV \quad ,$$

$$= \vec{a}_1 \int_0^1 r^4\, dr + i\vec{a}_2 \int_0^1 r^4\, dr \quad ,$$

$$= \frac{8\pi}{15}\ \vec{e}_+ \quad . \tag{4.35}$$

In physical variables this corresponds to an angular momentum

$$\frac{8\pi\rho a^5}{15}\ \vec{\omega}_1 \quad ,$$

which is just that of a homogeneous rigid sphere rotating with an angular velocity $\vec{\omega}_1$. This was to be expected, since $\vec{v}_o$ is the velocity field for infinite viscosity, or equivalently, rigid body motion. The first correction due to the liquid nature of the sphere appears in $\vec{\ell}_1$ which is

$$\vec{\ell}_1 = \int (-rv_{1\varphi}\vec{e}_\theta + rv_{1\theta}\vec{e}_\varphi)\, dV \quad ,$$

$$= -\frac{i\omega}{10}\ (\vec{a}_1 + ia_2) \int_o^1 r^4 (r^2-1) dr \quad ,$$

$$= \frac{8\pi i\omega}{525}\ \vec{e}_+ \quad . \tag{4.36}$$

At first sight, this term is similar to $\vec{\ell}_o$ since it is directed along $\vec{e}_+$. In the expression for $\vec{\ell}_o$, however, the coefficient in front of $\vec{e}_+$ is real, whereas in this instance it is imaginary. If we keep in mind the relationship between the complex angular momentum and the corresponding physical variable, as given by Eq. (4.31), it becomes evident that factors of $-i\omega$ are to be interpreted as time derivatives so that $\vec{\ell}_1$ corresponds to an actual physical angular momentum of

$$-\ \frac{8\pi\rho a^5}{525\epsilon^2}\ \frac{1}{\omega_o}\ \frac{d\vec{\omega}_1}{dt} \quad .$$

Finally, we evaluate $\vec{\ell}_2$ which is given by

$$\vec{\ell}_2 = \int (-rv_{2\varphi}\vec{e}_\theta + rv_{2\theta}\vec{e}_\varphi)\,dV \quad ,$$

$$= -(\vec{a}_1 + i\vec{a}_3)\ \frac{\omega}{2520}\int_0^1 r^3(7r^2-3)(r^2-1)dr$$

$$+\ \frac{\omega^2}{1400}\int_0^1 r^3(5r^2-9)(r^2-1)dr \quad ,$$

$$= \frac{\pi}{2520}\left[\frac{\omega}{9} - \frac{13\omega^2}{5}\right]\vec{e}_+ \quad . \tag{4.37}$$

In translating this result into physical terms, we run into trouble. The term proportional to ω^2, which comes from the toroidal velocity, presumably results from a double application of $-i\omega$, and so represents a second derivative with respect to time. The term proportional to ω, however, lacks the i which would make it a time derivative. This term arises from the poloidal velocity derived from S_2, the first non-zero term of this type. To interpret this term we examine the significance of $\omega\vec{e}_+$. The angular momentum associated with such a term is, according to Eq. (4.31)

$$\rho a^5 \omega_1 \frac{\Omega}{\omega_0}\ [\vec{e}_x \cos\Omega t + \vec{e}_y \sin\Omega t] \quad ,$$

which is a vector parallel to ω_1. The second correction to the angular momentum is thus

$$\frac{\pi\rho a^5}{2520\epsilon^4}\ \frac{\Omega}{9\omega_0}\ \vec{\omega}_1 + \frac{13}{5\omega_0^2}\ \frac{d^2\vec{\omega}_1}{dt^2} \quad .$$

The time derivative terms, of course, have an alternate interpretation, since

$$\frac{d\vec{\omega}_1}{dt} = \vec{\Omega} \times \vec{\omega}_1 \quad .$$

With the terms written in this way, the angular momentum of the fluid sphere, up to terms of order ϵ^{-4}, is

$$\vec{L} = \frac{8\pi\rho a^5}{15}\left[\vec{\omega}_R - \frac{\vec{\Omega}\times\vec{\omega}_R}{35\epsilon^2\omega_o} + \frac{1}{1344\epsilon^4\omega_o^{\,2}}\;\frac{(\vec{\Omega}\cdot\vec{\omega}_R)}{9} + \frac{13}{5}\,\vec{\Omega}\times(\vec{\Omega}\times\vec{\omega}_R)\right] \tag{4.38}$$

In this equation $\vec{\omega}_o$ and $\vec{\omega}_1$ have been eliminated in favor of the total angular velocity of the shell, $\vec{\omega}_R$. The combination of terms $\epsilon^2\omega_o$ which appears in the denominator is, by the definition of ϵ^2, the same as ν/a^2, so that the presence of ω_o in Eq. (4.38) is only apparent. The term arising from the solenoidal velocity, when written in this form, is seen to be a quadratic function of $\vec{\omega}_R$. Thus an effective moment of inertia for the fluid sphere cannot be strictly defined. If only the first correction is retained, then the moment of inertia tensor for the sphere is

$$I = \frac{8\pi\rho a^5}{15}\left[1 - \frac{a^2}{35\nu}\;\vec{\Omega}\times\right] . \tag{4.39}$$

Another dynamical quantity of interest is W, the total work done by the shell on the fluid per unit time. This is most easily evaluated from the expression

$$W = \vec{\omega}_R\cdot\frac{d\vec{L}}{dt} , \tag{4.40}$$

which states that the power expended is equal to the torque times the angular velocity about the direction of the torque. Keeping only the first correction, we find from Eq. (4.38) that

$$\frac{d\vec{L}}{dt} = \frac{8\pi\rho a^5}{15} \vec{\Omega} \times \vec{\omega}_R - \frac{a^2}{35\nu} \vec{\Omega} \times (\vec{\Omega} \times \omega_R) \quad , \tag{4.41}$$

so that

$$W = -\frac{8\pi\rho a^7}{525\nu} \vec{\omega}_R \cdot [\vec{\Omega} \times (\vec{\Omega} \times \vec{\omega}_R)] \quad ,$$

$$= \frac{8\pi\rho a^7}{525\nu} (\vec{\omega}_R \times \vec{\Omega})^2$$

$$= \frac{8\pi\rho a^7 \omega_1 \Omega^2}{525\nu} \quad . \tag{4.42}$$

4.4 Another Method of Iteration

In this section we shall follow the approach that was suggested in Eq. (4.27). Since the procedure for obtaining higher order approximations is basically the same as that outlined in Section 4.1, differing from it only in the complexity of the radial dependence, we shall obtain only the zeroth order approximation. This is equivalent to neglecting the operator A in the equations of motion. It should be noted that the term $A\vec{v}$ represents what in physical variables was a term $2\vec{\omega}_o \times \vec{u}$, that is, the Coriolis force. If the term proportional to A is omitted, the equation of motion is

$$\epsilon^2 \nabla^2 \vec{v} + i(\omega - 1)\vec{v} = \nabla Q \quad . \tag{4.43}$$

This equation is easily split into toroidal and poloidal parts, which are uncoupled. The equations for the scalars T and S are

$$\epsilon^2 \nabla^2 T + i(\omega - 1)T = 0 \quad , \tag{4.44}$$

and

$$\epsilon^2 \nabla^2 S + i(\omega - 1) S = \chi \quad , \tag{4.45}$$

where

$$\nabla^2 \chi = 0 \quad .$$

The boundary conditions on S and its normal derivative are homogeneous, so that $S = 0$, $\chi = 0$ is the obvious solution to Eq. (4.45).

T must be equal to $\sin\theta$ at $r = 1$, so that setting

$$T = f(r) \sin\theta \quad ,$$

we obtain the following differential equation for f:

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \left[\frac{i(\omega-1)}{\epsilon^2} - \frac{2}{r^2}\right] f = 0 \quad . \tag{4.46}$$

The non-singular solution to this equation is a spherical Bessel function or order 1, so that T becomes

$$T = \frac{j_1\left(\frac{r}{\epsilon}\sqrt{i(\omega-1)}\right)}{j_1\left(\frac{1}{\epsilon}\sqrt{i(\omega-1)}\right)} \sin\theta \quad . \tag{4.47}$$

It should be mentioned that the zeros of these functions lie along the real axis, so that T is well-behaved.

If ϵ is very large compared to $|\omega-1|^{\frac{1}{2}}$ the power series expansion for the Bessel function can be used and we then recover part of the solution obtained in Section 4.1. Of more interest, however, is the opposite case, when ϵ is much smaller than $|\omega-1|^{\frac{1}{2}}$, for, in that case, the argument of the Bessel functions in Eq. (4.47) is very large in absolute magnitude. We shall demonstrate that under these circumstances T is negligible except in a small region close to the surface of the sphere, the extent of this region being determined by $\epsilon / |\omega-1|^{\frac{1}{2}}$. To take a definite case, we assume that $(\omega-1)$ is positive and set

$$\frac{\omega-1}{\epsilon^2} = k^2 \quad ,$$

in which case Eq. (4.46) becomes

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{df}{dr} \right) + \left[ik^2 - \frac{2}{r^2} \right] f = 0 \quad . \tag{4.48}$$

The complex conjugate of this equation is

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{df^*}{dr} \right) - \left[ik^2 + \frac{2}{r^2} \right] f^* = 0 \quad . \tag{4.49}$$

Multiplying the first of these equations by $r^2 f^*$ and the second by $r^2 f$ and adding, we obtain

$$f^* \frac{d}{dr}\left(r^2 \frac{df}{dr} \right) + f \frac{d}{dr}\left(r^2 \frac{df^*}{dr} \right) - 2ff^* - 0 \quad .$$

Now

$$\frac{d}{dr}\left(r^2 \frac{d}{dr} (ff^*) \right) = f^* \frac{d}{dr}\left(r^2 \frac{df}{dr} \right) + f \frac{d}{dr}\left(r^2 \frac{df^*}{dr} \right) + 2r^2 \frac{df}{dr} \frac{df^*}{dr} \quad ,$$

so that

$$\frac{d}{dr}\left(r^2 \frac{dM}{dr} \right) = 2M + 2r^2 \left| \frac{df}{dr} \right|^2 \quad , \tag{4.50}$$

where

$$M = |f|^2 \quad .$$

Equation (4.50) can be put into the form

$$\frac{dM}{dr} = \frac{2}{r^2} \int \left(M + r^2 \left| \frac{df}{dr} \right|^2 \right) dr \quad . \tag{4.51}$$

Since M is by definition, a positive quantity, it follows that

$$\frac{dM}{dr} \geqslant 0 \quad ,$$

so that M is a non-decreasing function. We shall now show that M drops sharply as r decreases from 1. To do this, we need the asymptotic form of the spherical Bessel function for larges values of its argument. The required expression, taking into account the phase of the argument, is

$$j_1(kr\sqrt{i}) \rightarrow \frac{e^{(1-i)kr/\sqrt{2}}}{(1+i)kr/\sqrt{2}} \quad .$$

Adjusting the value of f to be 1 at r = 1, the corresponding expression for f is

$$f \rightarrow \frac{e^{-(1-i)k(1-r)/\sqrt{2}}}{r} \quad ,$$

or

$$|f|^2 \rightarrow \frac{e^{-k(1-r)\sqrt{2}}}{r^2} \quad ,$$

which has an exponential decay away from the boundary with a decay length of $1/k\sqrt{2}$. Thus the real and imaginary parts of f are oscillating functions which decay away from the boundary. The wavelength and the decay length are both measured by the ratio $\epsilon\sqrt{2}/(\omega-1)^{\frac{1}{2}}$, or, equivalently, $[2\nu/a^2(\Omega-\omega_0)]^{\frac{1}{2}}$. If we had assumed $\omega-1$ to be a negative quantity the results would have been similar, since the only difference is that Eq. (4.46) is of the form of Eq. (4.48), that is, f is replaced by f^* throughout.

These results indicate that if the viscosity is small compared to the difference between the precession frequency and the projection of the rotation frequency parallel to it, the flow is confined to a small region close to the boundary. The criterion which determines the smallness of the viscosity in this context is

$$\nu << \frac{1}{2} a^2 |\Omega - \omega_o| \quad .$$

5. THE CASE OF LOW VISCOSITY

5.1 The Boundary Layer

The momentum equation was written in the preceding chapter in the form

$$\epsilon^2 \nabla^2 \vec{v} + i(\omega - 1)\vec{v} = A\vec{v} + \nabla Q \quad , \tag{4.1}$$

and an expansion in powers of ϵ^{-2} was developed. We shall now investigate the case in which the viscosity is small compared with the other dimensions of the system, that is, when $\epsilon << 1$. From Eq. (4.1) it can be seen that in this case the first term of the equation is negligible unless the velocity field undergoes very rapid changes so that the term $\nabla^2 \vec{v}$, when compared to the other terms of the equation is of the order of ϵ^{-2}. This will mean, if the rapid change is in one dimension, that the velocity must change significantly in a dimensionless distance of the order of ϵ. In those parts of the fluid where such changes are not present, however, the flow should be adequately represented by the inviscid equation

$$[i(\omega-1) - A]\, \vec{v} = \nabla Q \quad . \tag{5.1}$$

This is an algebraic equation for $\vec{v}$ in terms of ∇Q, and thus $\vec{v}$ can be found as a linear combination of the first derivatives of Q. Since $\vec{v}$ must satisfy the equation of continuity, a single second order partial differential equation can be obtained for Q. This procedure will be carried out in detail in the next section. For the present, we can obtain the same equation by setting $\epsilon = 0$ in Eq. (3.37) which gives

$$\nabla^2 Q - \frac{4}{(\omega-1)^2} \frac{\partial^2 Q}{\partial z^2} = 0 \quad . \tag{5.2}$$

Depending on the value of ω, this equation is either elliptic or hyperbolic. For the elliptic case, at least, we know that to find a unique Q, one boundary condition must be specified. We have, however, to satisfy three boundary conditions which are given in Eq. (3.21). Since it could happen only accidentally that a solution of (5.2) satisfies all three conditions, we arrive at two conclusions. First, it would appear that Eqs. (5.1) and (5.2) cannot describe the flow near the boundary, and second, we do not know what boundary condition to apply to these equations in the region where they adequately describe the flow. In this section we shall obtain suitable boundary layer equations for the flow in the region immediately adjacent to the surface of the cavity.

In Section 3.2, the equations of motion were given in spherical coordinates. We shall approximate Eqs. (3.19) and (3.20) with a new set valid in the vicinity of $r = 1$ where we expect rapid changes in the radial direction. A systematic way of arriving at the approximate equations is to define suitable boundary layer variables, and, after the equations have been written in terms of these new variables, to set $\epsilon = 0$. The only independent variable that must be redefined is the radial one, since we expect the radial changes to be rapid, taking place in a distance of the order of ϵ. We thus define a new variable ξ, given by

$$\xi = (1-r)/\epsilon \quad . \tag{5.3}$$

In terms of ξ, the radial derivatives become

$$\frac{\partial}{\partial r} = - \frac{1}{\epsilon} \frac{\partial}{\partial \xi} \quad , \tag{5.4}$$

thus emphasizing their importance.

A consistent set of equations is obtained by further redefining two of the dependent variables:

$$v_r^{\dagger} = v_r/\epsilon \quad , \tag{5.5}$$

and

$$Q^{\dagger} = Q/\epsilon \quad . \tag{5.6}$$

If these new variables are inserted into Eqs. (3.19) and (3.20), and ϵ is then set equal to zero, the following equations result:

$$-i(\omega - 1)v_\theta - 2 \cos \theta \, v_\varphi - \frac{\partial^2 v_\theta}{\partial \xi^2} = 0 \quad , \tag{5.7}$$

$$-i(\omega - 1)v_\varphi + 2 \cos \theta \, v_\theta - \frac{\partial^2 v_\varphi}{\partial \xi^2} = 0 \quad , \tag{5.8}$$

$$2 \sin \theta \, v_\varphi = - \frac{\partial Q^{\dagger}}{\partial \xi} \quad , \tag{5.9}$$

and

$$- \frac{\partial v_r^{\dagger}}{\partial \xi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \, v_\theta) + \frac{i v_\varphi}{\sin \theta} = 0 \quad . \tag{5.10}$$

The first two of these equations are the θ and φ components of the momentum equation, the third is the r component, and the last one is the equation of continuity. If Eqs. (5.7) and (5.8) are used to solve for v_θ and v_φ, $Q^{\dagger}$ can then be obtained from Eq. (5.9), and $v_r^{\dagger}$ from Eq. (5.10).

A simple way of solving Eqs. (5.7) and (5.8) is to define the combinations

$$v_+ = v_\theta + iv_\varphi \quad ,$$

and (5.12)

$$v_- = v_\theta - iv_\varphi \quad .$$

It should be remarked that since v_θ and v_φ are themselves complex, v_- is not the complex conjugate of v_+. By multiplying Eq. (5.8) by i and adding or subtracting Eq. (5.7) we obtain the uncoupled equations

$$\frac{\partial^2}{\partial \xi^2} v_\pm + i[(\omega - 1) \mp 2 \cos \theta] v_\pm = 0 \quad . \tag{5.13}$$

In terms of $v_\pm$, the boundary conditions given by Eq. (3.21) are

$$\left.\begin{aligned} v_+ &= i(1 - \cos \theta) \quad , \\ v_- &= i(1 + \cos \theta) \quad , \\ v_r &= 0 \quad , \end{aligned}\right\} \text{ at } r = 1 \quad . \tag{5.14}$$

From Eq. (5.13) we thus obtain

$$\begin{aligned} v_+ &= i(1 - \cos \theta) e^{-\lambda_1 \xi} \quad , \\ v_- &= i(1 + \cos \theta) e^{-\lambda_2 \xi} \quad ; \end{aligned} \tag{5.15}$$

where

$$\lambda_1^2 = -i[(\omega - 1) - 2 \cos \theta] \quad ,$$

$$\lambda_2^2 = -i[(\omega - 1) + 2 \cos \theta] \quad ,$$

and λ_1 and λ_2 are assumed to have a positive real part so that Eq. (5.15) represents solutions that decay towards the interior of the fluid. The boundary layer thickness is therefore given by the greater of the quantities $\epsilon/|\lambda_1|$ or $\epsilon/|\lambda_2|$, and this solution truly represents a boundary layer provided $|\lambda_1|$ and $|\lambda_2|$ do not become as small as ϵ. It can be seen, however, that in the range of frequencies such that $(\omega-1)^2 < 4$, there will be one value of θ at which λ_1 is actually zero and also one value of θ at which λ_2 is zero. In the vicinity of these circles, the boundary layer thus appears to become infinitely thick. This phenomenon was discovered by Bondi and Lyttleton in Ref. 22 for the case $\omega = 0$. We shall discuss these critical circles more fully in Section 5.3, and proceed here to the evaluation of $v_r^\dagger$ and $Q^\dagger$, with the understanding that the results obtained will have to be modified near the critical circles for ω such that $(\omega-1)^2 < 4$. For what may be termed fast precession, however, that is for $(\omega-1)^2 > 4$, the solution is satisfactory on the entire surface.

From Eq. (5.10) and the definitions of v_+ and v_-, we obtain

$$\frac{\partial v_r^\dagger}{\partial \xi} = \frac{1}{2\sin\theta}\left[(v_+ - v_-) + \frac{\partial}{\partial\theta}\left[\sin\theta(v_+ + v_-)\right]\right] ,$$

$$= \frac{1}{2\sin\theta}\left[(1+\cos\theta)v_+ - (1-\cos\theta)v_-\right] + \frac{1}{2}\frac{\partial}{\partial\theta}(v_+ + v_-) ,$$

$$= \frac{i}{2}\sin\theta\,(e^{-\lambda_1\xi} - e^{-\lambda_2\xi}) + \frac{1}{2}\frac{\partial}{\partial\theta}(v_+ + v_-) .$$

This equation must be integrated subject to the condition that $v_r^\dagger = 0$ at $\xi = 0$. The result of the integration is

$$v_r^\dagger = \frac{i}{2}\sin\theta\left[\frac{1-e^{-\lambda_1\xi}}{\lambda_1} - \frac{1-e^{-\lambda_2\xi}}{\lambda_2}\right]$$

$$+ \frac{i}{2}\frac{\partial}{\partial\theta}\left[(1-\cos\theta)\frac{(1-e^{-\lambda_1\xi})}{\lambda_1} + (1+\cos\theta)\frac{(1-e^{-\lambda_2\xi})}{\lambda_2}\right] ,$$

$$= i\sin\theta\left[\frac{1-e^{-\lambda_1\xi}}{\lambda_1} - \frac{1-e^{-\lambda_2\xi}}{\lambda_2}\right]$$

$$+ \frac{i}{2}(1-\cos\theta)\frac{(\lambda_1\xi+1)e^{-\lambda_1\xi}-1}{\lambda_1^2}\frac{d\lambda_1}{d\theta} + \frac{i}{2}(1+\cos\theta)\frac{\lambda_2\xi+1)e^{-\lambda_2\xi}-1}{\lambda_2^2}\frac{d\lambda_2}{d\theta} .$$

Now,

$$\frac{d\lambda_1}{d\theta} = -\frac{i\sin\theta}{\lambda_1} ,$$

and

$$\frac{d\lambda_2}{d\theta} = \frac{i\sin\theta}{\lambda_2} ,$$

so that

$$v_r^\dagger = i\sin\theta\left(\left[\frac{1-e^{-\lambda_1\xi}}{\lambda_1} - \frac{i}{2}(1-\cos\theta)\frac{(\lambda_1\xi+1)e^{-\lambda_1\xi}-1}{\lambda_1^3}\right]\right.$$

$$\left. - \left[\frac{1-e^{-\lambda_2\xi}}{\lambda_2} - \frac{i}{2}(1+\cos\theta)\frac{(\lambda_2\xi+1)e^{-\lambda_2\xi}-1}{\lambda_2^3}\right]\right) . \qquad (5.16)$$

This expression goes to infinity as $1/\lambda$ as either of the λ's approaches zero.

The value of $v_r^\dagger$ at the edge of the boundary layer is of interest. This is obtained by setting $\xi = \infty$:

$$v_r^\dagger(\infty) = i\sin\theta\left[\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{i}{2}\left(\frac{1-\cos\theta}{\lambda_1^3} - \frac{1+\cos\theta}{\lambda_2^3}\right)\right] ,$$

$$= \frac{\sin\theta}{\lambda_1^3}\left[\omega - \frac{3}{2}(1+\cos\theta)\right] - \frac{\sin\theta}{\lambda_2^3}\left[\omega - \frac{3}{2}(1-\cos\theta)\right] \quad . \qquad (5.17)$$

This expression gives us the required boundary condition for the inviscid flow problem, since the boundary layer solution at the edge of the boundary layer must match the inviscid solution at the boundary. Thus the inviscid solution must satisfy the condition

$$v_r = \epsilon v_r^{\dagger}(\infty) \qquad (5.18)$$

at $r = 1$. Of course, there is no guarantee that the solution of the inviscid equations, subject to this boundary condition, will also satisfy the additional conditions that v_θ and v_φ are zero at the boundary which are required to match the two solutions completely. Since the equations of motion are linear, however, it follows that if the inviscid solution for v_r is of the order of ϵ at the boundary, then the corresponding values of v_θ and v_φ will also be of that order, and the boundary conditions for the whole problem are thus satisfied to within a correction which is of order ϵ compared with the actual condition. This is entirely consistent with the boundary layer approximation.

As a final step in this discussion of the boundary layer, we shall evaluate $Q^{\dagger}$. This is obtained from Eq. (5.9) which states that

$$\frac{\partial Q^{\dagger}}{\partial \xi} = i\sin\theta\,(v_+ - v_-) \quad ,$$

so that

$$Q^{\dagger} = \sin\theta\left[\frac{(1-\cos\theta)}{\lambda_1}\, e^{-\lambda_1 \xi} - \frac{(1+\cos\theta)}{\lambda_2}\, e^{-\lambda_2 \xi}\right] . \qquad (5.19)$$

An arbitrary function of θ can be added to this expression so that this relation imposes no restrictions on the inviscid solution.

The results obtained so far indicate that there are really two boundary layers occurring in this problem; thus $Q^\dagger$, for example, has two different decay lengths associated with it. In some regions one length is more important than the other; if for some value of θ $|\lambda_1| >> |\lambda_2|$, then the term involving λ_1 is important close to the boundary, while the term involving λ_2 would still be of significance at distances from the boundary at which the other term has already become negligible. To conform to the usual idea of a boundary layer, we shall speak of the boundary layer thickness as the decay length of the layer that survives the furthest from the boundary. The thickness is then given by

$$\delta(\theta) = \frac{\epsilon}{[\,|(|\omega-1| - 2|\cos\theta|)|\,]^{\frac{1}{2}}} \quad . \qquad (5.20)$$

For some values of θ this is the thickness associated with the terms involving λ_1 and for others that associated with terms involving λ_2. It is only in v_+ and v_- that these two scales are separated.

We have seen that for some values of the frequency the boundary layer thickness, as given by Eq. (5.20), may become infinite. We may distinguish between the rapidly precessing flows characterized by the inequality

$$(\omega-1)^2 > 4 \qquad \text{(fast precession)}$$

and the slowly precessing flows for which

$$(\omega-1)^2 < 4 \qquad \text{(slow precession)} \quad .$$

It is only in the latter flows that the critical circles occur. The circle on the surface of the sphere for which $\cos\theta = \frac{1}{2}(\omega-1)$ will be called C_+ since it is associated with the boundary layer that goes with v_+, and similarly the circle for which $\cos\theta = -\frac{1}{2}(\omega-1)$ will be denoted by C_-. These two circles are symmetrically located with respect to the equator of the sphere.

In Section 5.3 we shall discuss the difficulties that arise in the boundary layer treatment in the vicinity of the critical circles. We shall find that they can be dealt with by a more careful method of approximation. Before proceeding to that step, however, we shall explore some aspects of the inviscid flow problem which will bring out a peculiar relationship that exists between the boundary layer problem and the inviscid flow problem.

5.2 The Inviscid Equations

The equations that are obtained by setting $\epsilon = 0$ are most conveniently discussed in cylindrical coordinates rather than in spherical coordinates. In Section 3.3, the equations of motion were given in terms of the cylindrical coordinates (ρ, φ, z). If we set $\epsilon = 0$, in Eq. (3.25) we obtain

$$i(\omega-1)v_\rho + 2v_\varphi = \frac{\partial Q}{\partial\rho} \quad , \tag{5.21a}$$

$$i(\omega-1)v_\varphi - 2v_\rho = \frac{iQ}{\rho} \quad , \tag{5.21b}$$

and

$$i(\omega-1)v_z = \frac{\partial Q}{\partial z} \quad . \tag{5.21c}$$

In addition, we have the equation of continuity

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{i v_\varphi}{\rho} + \frac{\partial v_z}{\partial z} = 0 \quad . \tag{3.26}$$

As was pointed out earlier, the first three equations are a set of linear algebraic equations for the components of $\vec{v}$, and can be easily solved. From Eqs. (5.21a) and (5.21b) we obtain

$$v_\rho = \frac{i(\omega-1)\frac{\partial Q}{\partial\rho} - \frac{2iQ}{\rho}}{4-(\omega-1)^2} \quad , \tag{5.22}$$

and

$$v_z = -\frac{i}{\omega-1}\frac{\partial Q}{\partial z} \quad . \tag{5.24}$$

If these expressions are substituted into the equation of continuity, we obtain a single equation involving Q:

$$\frac{\partial^2 Q}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial Q}{\partial\rho} - \frac{Q}{\rho^2} + \frac{(\omega-1)^2-4}{(\omega-1)^2}\frac{\partial^2 Q}{\partial z^2} = 0 \quad , \tag{5.25}$$

which is just Eq. (5.2) written out in terms of cylindrical coordinates.

Equation (5.25) can be elliptic or hyperbolic depending on whether $(\omega-1)^2-4$ is positive or negative. We see that the inviscid problem also shows a division between fast and slow precession. In this latter case, when the equation is hyperbolic, the characteristics are given by

$$dz^2 - \frac{4-(\omega-1)^2}{(\omega-1)^2}\, d\rho^2 = 0 \quad ,$$

or

$$\frac{d\rho}{dz} = \pm\frac{(\omega-1)}{[4-(\omega-1)^2]^{\frac{1}{2}}} \quad . \tag{5.26}$$

This equation represents two families of right circular cones with axes along the z-axis. Althought Bondi and Lyttleton did find these cones for their case ($\omega = 0$) it seems to have escaped their notice that the circles of contact between the two cones that are tangent to the sphere are the critical circles C_+ and C_-. That this is so is easily demonstrated. The normal to the sphere through C_+ is inclined to the z-axis by an angle

$$\beta = \cos^{-1} \frac{\omega - 1}{2} \quad ,$$
$$= \tan^{-1} \frac{[4-(\omega-1)^2]^{\frac{1}{2}}}{(\omega-1)} \quad ,$$

and so is perpendicular to a cone whose generator is inclined from the z-axis with a slope

$$- \frac{(\omega-1)}{[4-(\omega-1)^2]^{\frac{1}{2}}} \quad ,$$

which is a cone belonging to one family of characteristics. Similarly the tangent cone touching C_- belongs to the other family of characteristics. This geometrical property connecting the characteristics of Eq. (5.25) to the critical circles is shown in Fig. 2 for a value of ω between 3 and 1. If ω lies between 1 and -1 the circles C_+ and C_- are interchanged since β is then an obtuse angle. Only the tangent cones have been drawn. Since the figure is a meridional cut through the sphere the critical circles appear as points and the tangent cones as lines. Figure 2 also shows schematically the boundary layer associated with v_+. For the sake of clarity the boundary layer has been given a thickness which is a considerable fraction of the

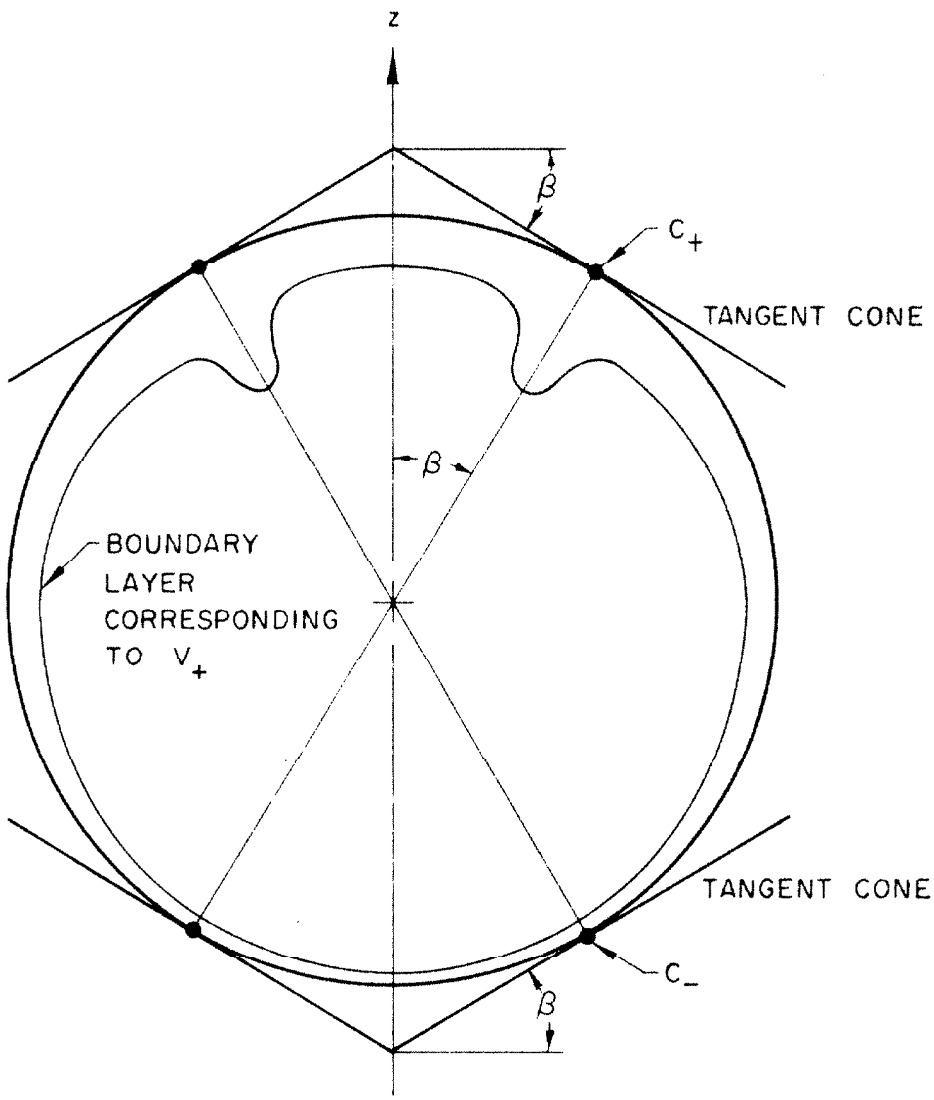

Figure 2. Geometry of the characteristic cones.

sphere's radius; this would presumably not be the case in the problem under consideration. The boundary layer associated with v_- is obtained by interchanging the positive and negative z-directions, while what we have called the boundary layer thickness $\delta(\theta)$ is obtained by repeating on the lower hemisphere the pattern that appears on the upper hemisphere.

Thus far, we have established some remarkable properties connecting the inviscid problem to its boundary layer counterpart. First, when $(\omega-1)^2 < 4$ the inviscid equation becomes hyperbolic and the critical circles appear with the indication that something has gone wrong in the boundary layer. Secondly, we have the geometric property just described connecting the characteristics of the inviscid problem to the critical circles in the boundary layer. These facts taken together strongly suggest that there is an intimate connection between the inviscid problem on the one hand and the associated boundary layer problem on the other. At this point it should be recalled that the original problem is represented by an elliptic system of equations so that it is only in the case in which the character of the equations changes in going from the viscous problem to the inviscid problem that difficulties arise in the boundary layer approximation. It would appear that this behavior is the key to an understanding of the critical circles.

Unfortunately, the topic of boundary layer approximations to elliptic equations which acquire real characteristics on dropping the higher order derivatives seems to have received little attention. An article by Visik and Lyusternik[31] deals with a problem having these features. The authors consider an equation having these properties in a region such that whole sections of the boundary are characteristic lines of the "inviscid" problem. Their example, however, is not sufficiently similar to our problem to warrant extensive discussion. In their closing remarks, the authors mention the problem of the inviscid characteristics becoming tangent to the boundary at a point as a matter worthy of future consideration.

5.3 Modified Boundary Layer Equations

In this section we shall reconsider the assumptions that led to the boundary layer equations obtained in Section 5.1. To see where the assumptions may have been inaccurate we return to the equation that was derived in that section

$$\frac{\partial^2 v_\pm}{\partial \xi^2} + 2i\,(\cos\beta \mp \cos\theta) v_\pm = 0 \quad , \qquad (5.13)$$

where $\cos\beta = \frac{1}{2}(\omega - 1)$. If we consider only v_+ for the moment, it is evident that according to this equation

$$\frac{\partial^2 v_+}{\partial \xi^2} = 0$$

on C_+, which immediately leads to an infinitely thick boundary layer.

[31] M. I. Viski and L. A. Lyusternik, "Regular Degeneration and Boundary Layer for Linear Differential Equations with Small Parameter," American Mathematical Society Translations, Series 2, 20 pp. 239-364 (Cushing-Malloy, Ann Arbor 1962). The relevant section starts in p. 301.

Now, this equation is evidently false; Eq. (5.13) was obtained by dropping certain terms which were assumed to be small compared to the ones retained. This assumption breaks down in the vicinity of C_+ since the term

$$(\cos\beta - \cos\theta)v_+$$

becomes very small there. Therefore, at least near C_+, the neglected terms can become as important or more important than the above term. Similar remarks apply to the equation for v_- in the vicinity of C_-. We must therefore rederive Eq. (5.13), paying closer attention to the terms that may become important. We shall still assume that the flow is confined to the vicinity of $r = 1$, so that radial changes are more important than angular ones. By this we mean that

$$\frac{\partial}{\partial r} >> \frac{\partial}{\partial \theta} \quad ,$$

but we shall no longer assume that

$$\epsilon^2 \frac{\partial^2}{\partial r^2} >> \frac{\partial}{\partial \theta} \quad .$$

While $\epsilon^2 \nabla^2$ may still be replaced by $\epsilon^2 \partial^2/\partial r^2$, we shall have to be more careful about discardingterms from the inviscid part of the equations of motion.

Equation (5.13) was obtained by combining the equations

$$\epsilon^2 (\nabla^2 \vec{v})_\theta = -2i \cos\beta v_\theta - 2\cos\theta v_\varphi + \frac{1}{r}\frac{\partial Q}{\partial \theta} \quad , \tag{3.19b}$$

and

$$\epsilon^2 (\nabla^2 \vec{v})_\varphi = -2i\cos\beta v_\varphi + 2\sin\theta\, v_r + 2\cos\theta v_\theta + \frac{iQ}{r\sin\theta} \quad . \tag{3.19c}$$

These equations can be approximated by

$$\epsilon^2 \frac{\partial^2 v_\theta}{\partial r^2} = -2i \cos \beta v_\theta - 2 \cos \theta v_\varphi + \frac{\partial Q}{\partial \theta} \quad ,$$

$$\epsilon^2 \frac{\partial^2 v_\varphi}{\partial r^2} = -2i \cos \beta v_\varphi + 2 \sin \theta \, v_r + 2 \cos \theta v_\theta + \frac{iQ}{\sin \theta} \quad .$$

The linear combinations which previously led to uncoupled equations for v_+ and v_- now give

$$\epsilon^2 \frac{\partial^2 v_+}{\partial r^2} = -2i(\cos \beta - \cos \theta) v_+ + 2i \sin \theta \, v_r + \frac{\partial Q}{\partial \theta} - \frac{Q}{\sin \theta} \quad , \qquad (5.27a)$$

and

$$\epsilon^2 \frac{\partial^2 v_-}{\partial r^2} = -2i(\cos \beta + \cos \theta) v_- - 2i \sin \theta \, v_r + \frac{\partial Q}{\partial \theta} + \frac{Q}{\sin \theta} \quad . \qquad (5.27b)$$

To eliminate v_r and Q, we have recourse to the r-component of the momentum equation and the equation of continuity. These are:

$$\epsilon^2 (\nabla^2 \vec{v})_r = -2i \cos \beta v_r - 2 \sin \theta v_\varphi + \frac{\partial Q}{\partial r} \quad , \qquad (3.19a)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{i v_\varphi}{r \sin \theta} = 0 \quad . \qquad (3.20)$$

Since $v_r = 0$ at $r = 1$, it can be assumed to be small compared to the tangential velocities, and so these equations may be approximated by

$$2 \sin \theta v_\varphi = \frac{\partial Q}{\partial r} \quad , \qquad (5.28)$$

and

$$\frac{\partial v_r}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \, v_\theta) + \frac{i v_\varphi}{\sin \theta} = 0 \quad . \qquad (5.29)$$

The variables v_r and Q can then be eliminated from Eq. (5.27a) by differentiating with respect to r:

$$\epsilon^2 \frac{\partial^3 v_+}{\partial r^3} + 2i(\cos\beta - \cos\theta)\frac{\partial v_+}{\partial r}$$

$$= 2i \sin\theta \frac{\partial v_r}{\partial r} + \frac{\partial^2 Q}{\partial r \partial\theta} - \frac{1}{\sin\theta}\frac{\partial Q}{\partial r} ,$$

$$= -2i \frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + 2v_\varphi + 2\frac{\partial}{\partial\theta}(\sin\theta\, v_\varphi) - 2v_\varphi$$

$$= -2i \frac{\partial}{\partial\theta}(\sin\theta\, v_+) \quad . \tag{5.30}$$

An equation for v_- is obtained in a similar manner from Eq. (5.27b):

$$\epsilon^2 \frac{\partial^3 v_-}{\partial r^3} + 2i(\cos\beta + \cos\theta)\frac{\partial v_-}{\partial r} = 2i\frac{\partial}{\partial\theta}(\sin\theta\, v_-) \quad . \tag{5.31}$$

Equations (5.30) and (5.31) replace Eq. (5.13). It should be noticed that Eq. (5.31) is transformed into Eq. (5.30) if θ is replaced by $\pi - \theta$. Moreover, the boundary conditions

$$\left.\begin{aligned} v_+ &= i(1 - \cos\theta) \\ v_- &= i(1 + \cos\theta) \end{aligned}\right\} \quad \text{at} \quad r = 1 \quad , \tag{5.14}$$

also transform into each other under this change of variables. It follows that

$$v_-(r, \theta) = v_+(r, \pi - \theta) \quad . \tag{5.32}$$

We can therefore deal with a single equation since once one of these variables is found the other can be obtained directly from it.

It is convenient to deal not with v_+ or v_- but with a variable v defined by

$$v = \frac{4i\, v_+ \sin\theta}{(\omega-1)[4-(\omega-1)^2]^{\frac{1}{2}}} , \qquad (5.33)$$

in terms of which Eq. (5.30) becomes

$$\epsilon^2 \frac{\partial^3 v}{\partial r^3} + 2i(\cos\beta - \cos\theta) \frac{\partial v}{\partial r} + 2i \sin\theta \frac{\partial v}{\partial\theta} = 0 , \qquad (5.34)$$

subject to the boundary condition

$$v = - \frac{4 \sin\theta(1-\cos\theta)}{(\omega+1)[4-(\omega-1)^2]^{\frac{1}{2}}} \qquad (5.35)$$

at $r = 1$. The numerical factors have been chosen to make $v = 1$ on C_+.

Since so many approximations have been made already, it would be highly desirable to solve Eq. (5.34) as it stands. Unfortunately, we were not able to do this, and even after further mutilation of this equation we found that we had to be content with an approximate solution. Before proceeding to that stage, however, certain interesting properties of Eq. (5.34) should be pointed out.

First of all, if the last term is dropped, we are back at Eq. (5.17) and the expressions for v_+ and v_- that result from it. By differentiating them with respect to θ, we can justify the neglect of that term for regions sufficiently removed from the critical circles. As the critical circles are approached, the second term in Eq. (5.34) becomes smaller, but the third term survives so that $\partial^2 v/\partial r^2$ does not become zero. Another interesting property of Eq. (5.34) is that the inviscid equation associated with it has real characteristics. The inviscid equation is obtained by setting $\epsilon = 0$:

$$\frac{\partial v}{\partial r} + \frac{\sin\theta}{\cos\beta - \cos\theta}\,\frac{\partial v}{\partial\theta} = 0 \quad .$$

The characteristics of this equation have a slope

$$\frac{dr}{d\theta} = \frac{\cos\beta - \cos\theta}{\sin\theta} \quad . \tag{5.36}$$

This slope is zero on $\theta = \beta$, so that the characteristic tangent to the sphere touches the sphere on C_+, as before.

The characteristics given by Eq. (5.36) are not the same as those of the original problem. This is due to the distortion in the co-ordinates that was introduced into the derivation of Eq. (5.34). In the approximation used the curvature of the geometry was neglected by setting $r = 1$ wherever it appeared as a multiplicative factor. This fault is minor, and could be corrected. What matters is that characteristics appear, and that the line of tangency is the critical circle.

Since the new boundary layer equation represents an improvement over the old principally in the vicinity of C_+, it would appear worthwhile to examine the equation near $\theta = \beta = \cos^{-1}\frac{\omega-1}{2}$. To this end we stretch the region of interest by defining a new set of variables

$$\begin{aligned} \xi &= \frac{1-r}{A\epsilon^{\mu}} \quad , \\ \eta &= \frac{\theta-\beta}{B\epsilon^{\nu}} \quad , \end{aligned} \tag{5.37}$$

where μ and ν are assumed positive but are left otherwise undetermined, and A and B are constants of order 1 which will also be fixed later. Keeping only the lowest order terms in ϵ, we have

$$\cos\theta = \cos\beta - B\epsilon^{\nu}\sin\beta\eta \quad ,$$

$$\frac{\partial}{\partial r} = -\frac{1}{A\epsilon^{\mu}}\frac{\partial}{\partial\xi} \quad ,$$

$$\sin\theta\frac{\partial}{\partial\theta} = \frac{\sin\beta}{B\epsilon^{\nu}}\frac{\partial}{\partial\eta} \quad ,$$

so that Eq. (5.34) becomes

$$\frac{\epsilon^{2-3\mu}}{A^3}\frac{\partial^3 v}{\partial\xi^3} + \frac{2iB}{A}\sin\beta\,\epsilon^{\nu-\mu}\eta\frac{\partial v}{\partial\xi} - \frac{2i}{B}\sin\beta\epsilon^{-\nu}\frac{\partial v}{\partial\eta} = 0 \ . \quad (5.38)$$

It is convenient to choose A and B in such a way that

$$2A^2B\sin\beta = 1 \quad ,$$

and

$$\frac{2A^3\sin\beta}{B} = 1 \quad ,$$

that is

$$A = (4\sin^2\beta)^{-1/5} \quad ,$$

$$B = 2\sin\beta(4\sin^2\beta)^{-3/5} \quad .$$

With this change, Eq. (5.38) becomes

$$\frac{\partial^3 v}{\partial\xi^3} + i\epsilon^{2\mu+\nu-2}\eta\frac{\partial v}{\partial\xi} - i\epsilon^{3\mu-\nu-2}\frac{\partial v}{\partial\eta} = 0 \quad . \qquad (5.39)$$

The relative importance of the terms in this equation can be enhanced or diminished by suitable choice of μ and ν. Thus, setting $2\mu + \nu - 2 = 0$, $3\mu - \nu - 2 > 0$, and then letting $\epsilon = 0$, we arrive at the equation

$$\frac{\partial^3 v}{\partial\xi^3} + i\eta\frac{\partial v}{\partial\xi} = 0 \quad ,$$

with the solution

$$v = e^{\xi\sqrt{-i\eta}} \quad , \tag{5.40}$$

which corresponds to Eq. (5.15). This expression cannot be valid near $\eta = 0$ since it has an infinite derivative with respect to η there, contradicting the assumption that permitted us to drop the last term. It would be desirable, however, to obtain an expression for v that has this behavior for large values of η, so that it can be matched to the solutions previously obtained. This dictates the choice of μ and ν to be such that all the terms in Eq. (5.39) survive, that is

$$2\mu + \nu - 2 = 0 \quad ,$$

$$3\mu - \nu - 2 = 0 \quad ,$$

or

$$\mu = 4/5 \quad , \qquad \nu = 2/5 \quad . \tag{5.41}$$

With these values of μ and ν, Eq. (5.39) takes the form

$$\frac{\partial^3 v}{\partial \xi^3} + i\eta \frac{\partial v}{\partial \xi} - i \frac{\partial v}{\partial \eta} = 0 \quad . \tag{5.42}$$

An equation equivalent to this one was obtained by Stewartson and Roberts[32] in connection with the Maclaurin spheroid by applying the substitutions given in Eq. (5.37) directly to the equations of motion. In Ref. [27] they pointed out that the same equation was applicable to the critical circles in the problem of the precessing spheroid. They limited themselves, however, to giving the asymptotic solution of

[32] P. Roberts and K. Stewartson, "On the Stability of a Maclaurin Spheroid of Small Viscosity," Astrophys. J. 137, 777 (1963).

Eq. (5.40) and inferring, from the method by which Eq. (5.39) was derived, that the boundary layer for the neighborhood of the critical circle given by

$$\theta = \beta + O(\epsilon^{2/5})$$

has a thickness

$$\delta = O(\epsilon^{4/5})$$

in contrast with a thickness of the order of ϵ which applies to the rest of the boundary layer.

While we were not able to solve Eq. (5.42) exactly, in the next section we shall obtain an approximate solution which strongly supports the above conclusion.

5.4 Approximate Treatment of the Critical Circles

The boundary layer problem in the vicinity of the critical circles has been reduced to the equation

$$\frac{\partial^3 v}{\partial \xi^3} + i\eta \frac{\partial v}{\partial \xi} - i \frac{\partial v}{\partial \eta} = 0 \quad , \tag{5.42}$$

subject to the conditions:

$$v = 1 \quad \text{on} \quad \xi = 0 \quad ,$$

$$v \to e^{-\xi\sqrt{-i\eta}} \quad \text{as} \quad |\eta| \to \infty \quad ,$$

$$v \to 0 \quad \text{as} \quad \xi \to \infty \quad .$$

The first condition is just the normalization of v at the critical circle. The second one is imposed to make v match the solutions far from the critical circles; in it the square root in the exponential is interpreted as having a positive real part. The third condition is required to make v have a boundary layer character.

The most successful approach that we found for the solution of this problem is to make the substitution

$$v = e^{\varphi} \quad , \tag{5.43}$$

so that

$$\frac{\partial v}{\partial \xi} = e^{\varphi} \frac{\partial \varphi}{\partial \xi} \quad ,$$

$$\frac{\partial^3 v}{\partial \xi^3} = e^{\varphi} \left[\left(\frac{\partial \varphi}{\partial \xi} \right)^3 + 3 \frac{\partial \varphi}{\partial \xi} \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^3 \varphi}{\partial \xi^3} \right] \quad .$$

The next step is to neglect the terms in $\partial^3 v / \partial \xi^3$ involving second and third derivatives of φ . This approximation is easier to accept if we leave the equation for v in the form

$$\frac{\partial^3 v}{\partial \xi^3} + i \, \epsilon^{2\mu+\nu-2} \eta \frac{\partial v}{\partial \xi} - i \, \epsilon^{3\mu-\nu-2} \frac{\partial v}{\partial \eta} = 0 \quad , \tag{5.39}$$

and use the substitution

$$v = \epsilon^{\varphi / \epsilon} \quad .$$

Then Eq. (5.39) becomes

$$\left(\frac{\partial \varphi}{\partial \xi} \right)^3 + 3\epsilon \frac{\partial \varphi}{\partial \xi} \frac{\partial^2 \varphi}{\partial \xi^2} + \epsilon^2 \frac{\partial^3 \varphi}{\partial \xi^3} + i \, \epsilon^{2\mu+\nu} \eta \frac{\partial \varphi}{\partial \xi} - i \, \epsilon^{3\mu-\nu} \frac{\partial \varphi}{\partial \eta} = 0 \quad .$$

If we now set μ and ν both equal to zero, and then take the limit as ϵ goes to zero, the higher derivatives of φ drop out. It is probably preferable to drop these terms without going through this process so as not to lose sight of the approximation involved. With the omission of these terms, the equation for φ is

$$\left(\frac{\partial\varphi}{\partial\xi}\right)^3 + i\eta\frac{\partial\varphi}{\partial\xi} - i\frac{\partial\varphi}{\partial\eta} = 0 \quad . \tag{5.44}$$

Since this equation does not contain ξ explicitly, a complete integral (involving two arbitrary constants) is obtained by setting

$$\frac{\partial\varphi}{\partial\xi} = a \quad ,$$

which substituted into Eq. (5.44) gives

$$\frac{\partial\varphi}{\partial\eta} = ia^3 + a\eta \quad .$$

Combining these two expressions, the result is

$$\varphi = a\xi - ia^3\eta + \frac{1}{2}a\eta^2 + b \quad . \tag{5.45}$$

To obtain a general integral (involving one arbitrary function) we employ Jacobi's method which consists of setting $b = f(a)$ and then determining a by requiring that $\partial\varphi/\partial a = 0$.

Thus, we obtain:

$$\begin{aligned} f'(a) &= 3ia^2\eta - \xi - \frac{1}{2}\eta^2 \quad , \\ \varphi &= a\xi - ia^3\eta + \frac{1}{2}a\eta^2 + f(a) \quad . \end{aligned} \tag{5.46}$$

The first of these expressions defines a function $a(\xi, \eta)$ implicitly in terms of an arbitrary function $f(a)$. Once $a(\xi, \eta)$ is known, the second expression gives $\varphi(\xi, \eta)$. That this expression satisfies the differential equation is easily verified by substitution.

Our problem is to find a suitable function $f(a)$. The process here is one of guesswork since we do not know which of the conditions on v to apply. One condition is that at $\xi = 0$, v should equal 1, or $\varphi = 0$. If we apply this requirement, we have

$$f(a) = ia^3\eta - \frac{1}{2} a\eta^2 \quad ,$$

$$f'(a) = 3ia^3\eta - \frac{1}{2} \eta^2 \quad ;$$

where, since $a = a(0,\eta)$, η can be regarded as a function of a. Differentiating the first of these expressions, we obtain

$$f'(a) = 3ia^3\eta - \frac{1}{2} \eta^2 + (ia^3 - a\eta) \frac{d\eta}{da} \quad .$$

It follows that

$$\eta = ia^2 \quad ,$$

and hence

$$f(a) = -\frac{1}{2} a^5 \quad . \tag{5.47}$$

Putting this back into Eq. (5.46) we obtain

$$\frac{5}{2} a^4 + 3ia^2\eta - \left(\xi + \frac{1}{2} \eta^2\right) = 0 \quad ,$$

which is fortunately a quadratic equation for a^2. The roots are

$$a^2 = -\frac{3}{5} i\eta \pm \left[\frac{2}{5}\xi - \frac{4}{25}\eta^2\right]^{\frac{1}{2}} \quad . \tag{5.48}$$

With suitably chosen branches of the square roots involved, we finally obtain

$$\varphi = a\xi + \frac{1}{2} a\eta^2 - ia^3\eta - \frac{1}{2} a^5 \quad . \tag{5.49}$$

We must now determine whether the branches of the square roots in

Eq. (5.48) can be chosen in such a way that φ will have the correct behavior as $|\eta|$ approaches infinity:

$$\varphi \to -\xi \frac{(1-i)}{\sqrt{2}} \sqrt{\eta} \quad , \quad \text{for} \quad \eta \to \infty \quad ;$$

$$\varphi \to -\xi \frac{(1+i)}{\sqrt{2}} \sqrt{-\eta} \quad , \quad \text{for} \quad \eta \to -\infty \quad .$$

This dictates the form of a: Eq. (5.48) can be written as

$$a^2 = -\frac{3}{5} i\eta \pm \frac{2}{5} i\eta \left[1 - \frac{5\xi}{2\eta^2}\right]^{\frac{1}{2}} \quad ,$$

if the minus sign is chosen, then for $\eta^2 >> \xi$, a^2 will be given approximately by

$$a^2 \simeq -i\eta + \frac{i\xi}{2\eta} \quad ,$$

so that φ becomes approximately

$$\varphi \simeq a\left[\xi + \frac{1}{2}\eta^2 - i\eta\left(-i\eta + \frac{i\xi}{2\eta}\right) - \frac{1}{2}\left(-i\eta + \frac{i\xi}{2\eta}\right)^2\right] = a\xi\left[1 + \frac{\xi}{8\eta^2}\right] \quad .$$

To achieve the required form for φ, we must now take a as that square root of a^2 that has a negative real part. Thus we must have

$$a = -\frac{(1-i)}{\sqrt{2}}\left[\frac{3}{5}\eta + \frac{2}{5}\eta\left(1 - \frac{5\xi}{2\eta^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad , \qquad (\eta > 0)$$

or

$$a = -\frac{(1+i)}{\sqrt{2}}\left[-\frac{3}{5}\eta - \frac{2}{5}\eta\left(1 - \frac{5\xi}{2\eta^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad . \quad (\eta < 0) \qquad (5.50)$$

Equations (5.49) and (5.50) then give expressions for φ which satisfy both the boundary condition and the asymptotic requirements for large $|\eta|$. We would now like to find out what information this gives us regarding φ near $\eta = 0$, since this is the region where the previous expressions failed. Keeping track of the branch of the square root involved, we get, for $\eta^2 < \frac{2}{5}\xi$

$$a^2 = -\frac{3}{5}i\eta - \left(\frac{2}{5}\xi\right)^{\frac{1}{2}}\left(1 - \frac{2\eta^2}{5\xi}\right)^{\frac{1}{2}} ,$$

while a is still that square root which has a negative real part.

For $\eta^2 << \xi$,

$$a^2 \simeq -\frac{3}{5}i\eta + \left(\frac{2}{5}\xi\right)^{\frac{1}{2}} - \frac{2}{25}\frac{\eta^2}{\left(\frac{2}{5}\xi\right)^{\frac{1}{2}}} ,$$

while

$$a \simeq -\left(\frac{2}{5}\xi\right)^{\frac{1}{4}}\left[1 - \frac{3i}{10}\frac{\eta}{\left(\frac{2}{5}\xi\right)^{\frac{1}{2}}}\right] .$$

From these expressions we obtain

$$\varphi \simeq -2\left(\frac{2}{5}\xi\right)^{5/4}\left[1 - \frac{4}{5}\frac{\eta}{\left(\frac{2}{5}\xi\right)^{\frac{1}{2}}}\right] .$$

If we now express ξ and η in terms of r, θ, and ϵ we obtain

$$v \simeq \exp\left\{-2\left(\frac{2}{5}\right)^{5/4}\frac{(1-r)^{5/4}}{\epsilon}(2\sin\beta)^{\frac{1}{2}}\left[1 - \frac{3i}{10}\frac{\theta - \beta}{(1-r)^{\frac{1}{2}}(2\sin\beta)^{\frac{1}{2}}}\right]\right\} \quad (5.51)$$

where this expression holds for

$$(\theta - \beta)^2 << 1 - r .$$

This result shows that the boundary layer near the critical circles dies off in a distance of the order of $\epsilon^{4/5}$, which is greater than the decay length of the order of ϵ which occurs in the rest of the boundary layer.

5.5 Validity of the Approximation

The approximation obtained in the preceding section fails if $\partial^2\varphi/\partial\xi^2$ or $\partial^3\varphi/\partial\xi^3$ become large. Unfortunately, this does happen, for along the parabola $\xi = \frac{2}{5}\eta^2$ these derivatives are infinite. This is easily established, since

$$\frac{\partial\varphi}{\partial\xi} = a \quad ,$$

so that

$$\frac{\partial^2\varphi}{\partial\xi^2} = \frac{\partial a}{\partial\xi} \quad ,$$

and

$$\frac{\partial^3\varphi}{\partial\xi^3} = \frac{\partial^2 a}{\partial\xi^2} \quad .$$

Now,

$$a^2 = -\frac{3}{5}\,i\eta - \frac{2}{5}\,i\eta\left[1 - \frac{5\xi}{2\eta^2}\right]^{\frac{1}{2}} \quad ,$$

so that both $\partial a/\partial\xi$ and $\partial^2 a/\partial\xi^2$ are infinite along the curve $\xi = \frac{2}{5}\eta^2$, where the square root term is zero. Thus, whereas we were previously in difficulties because $\partial v/\partial\eta$ became infinite along $\eta = 0$, we have now removed that trouble only to have a new one arise in the ξ derivatives along the parabola $\xi = \frac{2}{5}\eta^2$. Since this parabola lies in the region of the $\xi - \eta$ plane which is of interest to us, this difficulty cannot be ignored. Indeed, one may question whether Eq. (5.49), which gives φ in terms of ξ, η and a, is valid on both sides of this curve and can be extended across it by merely interpreting the square roots in a consistent manner. If this procedure is valid, it would be desirable to find a connection formula to join these two solutions smoothly.

We shall now give an argument, which although admittedly weak, suggests that this procedure is probably acceptable.

The equation for v is

$$\frac{\partial^3 v}{\partial \xi^3} + i\eta \frac{\partial v}{\partial \xi} - i \frac{\partial v}{\partial \eta} = 0 \quad . \tag{5.42}$$

Let us look for a solution of the form

$$v = e^{k\xi} g(\eta) \quad .$$

Then $g(\eta)$ satisfies

$$\frac{dg}{d\eta} = (k\eta - ik^3) g \quad ,$$

so that

$$g = C e^{\frac{1}{2} k\eta^2 - ik^3 \eta} \quad .$$

We might therefore attempt to represent v by a superposition of such solutions:

$$v = \int F(k) \exp\left[k\xi + \frac{1}{2} k\eta^2 - ik^3 \eta + f(k)\right] dk, \tag{5.52}$$

where the arbitrary function of k has been written as $F(k) \exp f(k)$. We can require that $f(k)$ include any exponential behavior so that $F(k)$ is no worse than some power of k. The integral is of course taken over some contour, which we are unable to determine.

Equation (5.52) is of the form

$$v = \int F(k) e^{\varphi(\xi, \eta, k)} dk \quad .$$

If it were desired to evaluate this integral by the method of steepest descents, we would set

$$\frac{\partial\varphi}{\partial k} = 0 \quad ,$$

and thus obtain a function $k(\xi, \eta)$. The approximate expression for v would then be

$$v = \frac{F(k)(2\pi)^{\frac{1}{2}}}{\left|\frac{\partial^2\varphi}{\partial k^2}\right|^{\frac{1}{2}}} e^{\frac{1}{2}i(\pi-\alpha)} e^{\varphi} \tag{5.53}$$

where $\alpha = \arg \frac{\partial^2\varphi}{\partial k^2}$. In this equation, k is a function of ξ and η, implicitly given by $\partial\varphi/\partial k = 0$.

This form of the method of steepest descents fails if $\partial\varphi/\partial k$ and $\partial^2\varphi/\partial k^2$ are both zero simultaneously. One then has to go back to the original integral and expand φ in a power series up to the first non-vanishing term in k, and then carry out the approximation.

The point of this discussion is that if one were to choose $f(k)$ in Eq. (5.52) equal to $-\frac{1}{2}k^5$, then Eq. (5.53) would be remarkably similar to what was obtained in the preceding section, since v would essentially be given by e^{φ}, with φ given by the same expression as before. Furthermore, setting

$$\varphi = k\xi + \frac{1}{2}k\eta^2 - ik^3\eta - \frac{1}{2}k^5 \quad ,$$

we have that

$$\frac{\partial\varphi}{\partial k} = \xi + \frac{1}{2}\eta^2 - 3ik^2\eta - \frac{5}{2}k^4 \quad ,$$

$$\frac{\partial^2\varphi}{\partial k^2} = -6ik\eta - 10k^3 \quad ,$$

so that $\partial\varphi/\partial k$ and $\partial^2\varphi/\partial k^2$ are both zero when $\xi = \frac{2}{5}\eta^2$. These considerations suggest that the expressions for φ obtained in

Section 5.4 are asymptotic forms of an integral like the one given in Eq. (5.52) with $f(k) = -\frac{1}{2}k^5$. Since neither $F(k)$ or the countour are known, there is not much point in carrying this argument further. The inference, however, is that a connection formula probably exists. While we are quite aware that qualitative evidence is not admissible in mathematical arguments, we have presented this discussion as supporting the plausibility of the approximate solution for v obtained is an accurate representation of the actual v.

5.6 The Interior Flow

In Section 5.2 the equations for the inviscid flow were solved to give the velocity components in terms of the variable Q, and a differential equation was obtained for Q. With a slight change of notation to simplify the expressions involved, the equation for Q is

$$\frac{\partial^2 Q}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial Q}{\partial \rho} - \frac{Q}{\rho^2} + \tanh^2\alpha \, \frac{\partial^2 Q}{\partial z^2} = 0 \quad , \qquad (5.54)$$

where a variable α has been defined such that

$$\cosh\alpha = \frac{\omega - 1}{2} \quad . \qquad (5.55)$$

This definition is suitable for $\omega > 3$, since α is then real. In Section 5.2 a real variable β was defined for $-1 < \omega < 3$ by the relation

$$\cos\beta = \frac{\omega - 1}{2} \quad .$$

The two relations are connected by

$$\alpha = i\beta \quad .$$

Rather than define a third real variable for $\omega < -1$, we shall use α throughout the elliptic range of Eq. (5.54) and β for the hyperbolic range.

The velocity components can also be expressed in terms of α. Of particular interest is the radial velocity, which is obtained from Eqs. (5.22) and (5.24) and is given by

$$\begin{aligned} rv_r &= \rho v_\rho + zv_z \quad , \\ &= \frac{i}{2} \operatorname{sech}\alpha \operatorname{cosech}^2\alpha\left[\cosh^2\alpha\rho \frac{\partial Q}{\partial \rho} - \cosh\alpha Q + \sinh^2\alpha\, z \frac{\partial Q}{\partial z}\right] \quad . \end{aligned} \tag{5.56}$$

The boundary condition that applies to the inviscid problem is

$$v_r(\text{inviscid}) = v_r(\infty) \quad , \tag{5.57}$$

at $r = 1$, where $v_r(\infty)$ is the value of v_r at the edge of the boundary layer. Thus Q is required to satisfy Eq. (5.54) subject to the boundary condition (5.57).

In the elliptic range, Eq. (5.54) can be transformed into Laplace's equation by simply substituting

$$\begin{aligned} \rho' &= \rho \sinh\alpha \quad , \\ z' &= z \cosh\alpha \quad . \end{aligned} \tag{5.59}$$

This transformation maps the sphere into a prolate ellipsoid, and the problem of finding Q reduces to the solution of Laplace's equation subject to a boundary condition involving Q and its derivatives. The problem can thus be solved, in principle, by expanding Q in solid harmonics.

In a recent paper, which appeared while this thesis was in preparation, Greenspan[33] has demonstrated that the equation for Q can be solved by separation of variables in prolate ellipsoidal coordinates. The procedure is to transform from the coordinates (ρ', z') to prolate ellipsoidal coordinates (λ, μ) in such a way that the surface of the sphere transforms into a surface of constant λ. The required transformation is

$$\begin{aligned} \rho' &= \sqrt{(\lambda^2-1)(1-\mu^2)} \\ z' &= \lambda\mu \quad . \end{aligned} \tag{5.60}$$

The surface of the sphere is then given by $\lambda = \cosh\alpha$, while μ becomes equal to $\cos\theta$. In terms of λ and μ the derivatives involved in Eq. (5.56) are:

$$\rho \frac{\partial}{\partial\rho} = \frac{(\lambda^2-1)(1-\mu^2)}{\lambda^2 - \mu^2}\left[\lambda \frac{\partial}{\partial\lambda} - \mu \frac{\partial}{\partial\mu}\right] ,$$

$$z \frac{\partial}{\partial z} = \frac{\lambda\mu}{\lambda^2-\mu^2}\left[\mu(\lambda^2-1) \frac{\partial}{\partial\lambda} + \lambda(1-\mu^2) \frac{\partial}{\partial\mu}\right] ,$$

so that the radial velocity is given by

$$\begin{aligned} rv_r = \frac{i \operatorname{sech}\alpha \operatorname{cosech}^2\alpha}{2(\lambda^2-\mu^2)} \Big[& \lambda(\lambda^2-1)(\cosh^2\alpha-\mu^2) \frac{\partial Q}{\partial\lambda} \\ & -\mu(1-\mu^2)(\lambda^2-\cosh^2\alpha) \frac{\partial Q}{\partial\mu} \\ & -\cosh\alpha(\lambda^2-\mu^2)Q \Big] \quad . \end{aligned}$$

It can be seen that on the boundary the coefficient of $\partial Q/\partial\mu$ is zero,

[33] H. P. Greenspan, "On the Transient Motion of a Contained Rotating Fluid," J. Fluid Mech. 20, 673 (1964).

so that the boundary condition involves only Q and $\partial Q/\partial\lambda$. This feature makes the problem separable, since in terms of λ and μ the boundary condition becomes:

$$\frac{\partial Q}{\partial\lambda} - \operatorname{cosech}^2\alpha\, Q = -2iv_r(\infty) \quad , \qquad (5.61)$$

at $\lambda = \cosh\alpha$.

The non-singular separated solutions to Laplace's equation in prolate ellipsoidal coordinates are[34]:

$$P_n^1(\lambda)P_n^1(\mu)$$

so that Q can be expanded into a sum

$$Q = \sum A_n P_n^1(\lambda)P_n^1(\mu) \quad . \qquad (5.62)$$

Similarly $v_r(\infty)$ can be expanded into

$$v_r(\infty) = \sum B_n P_n^1(\cos\theta) \quad ,$$

$$= \sum B_n P_n^1(\mu) \quad , \qquad (5.63)$$

since $\mu = \cos\theta$ on the surface. The coefficients A_n can then be found in terms of the B_n

$$A_n\left[\frac{dP_n^1(\cosh\alpha)}{d(\cosh\alpha)} - \frac{P_n^1(\cosh\alpha)}{\cosh^2\alpha - 1}\right] = -2i\,B_n \quad . \qquad (5.64)$$

[34] Morse and Feshbach, Methods of Theoretical Physics (McGraw Hill, New York, 1953), p. 1285.

The function

$$F_n(x) = \frac{dP_n^1}{dx} + \frac{P_n^1}{1-x^2} \quad ,$$

which appears as the coefficient of A_n in Eq. (5.64) can be rewritten (with a change in notation from Greenspan's article) in the form

$$F_n(x) = \frac{1+x}{1-x} \frac{d}{dx}\left[\frac{P_n^1(x)}{1-x}\right] \quad .$$

Since all the zeros of $P_n^1(x)$ are in $-1 < x < 1$, the zeros of $F_n(x)$ will similarly be in this interval, and thus the coefficient of A_n does not vanish provided $(\omega-1)^2 > 1$. In the hyperbolic range of Eq. (5.54), however, the transformation equations from (ρ, z) to (λ, μ) map the surface of the sphere into the surface $\lambda = \cos\beta$, so that for each n there are several roots β of $F_n(\cos\beta)$. These frequencies are resonances in the true sense of the word, since the corresponding A_n then become infinite. Moreover, they represent inviscid modes of fluid motion in which the radial velocity is identically zero. The existence of these modes makes it impossible to satisfy the inviscid flow equations subject to a specified radial velocity on the boundary. The only remedy to the situation is to change the boundary conditions of the boundary layer equations in such a way that the radial velocity at the edge of the boundary layer becomes zero. This can be done by adding to the interior flow resonant solutions of order 1. Thus the interior flow at the resonant frequencies is of order 1 while it is only of order ϵ at other frequencies.

5.7 The Case $\omega = 3$

The frequency $\omega = 3$ is one of the dividing points between rapidly and slowly precessing flows. The equations of motion are particularly simple at this frequency and they will be considered here. The expressions for the velocity components in terms of Q given in Section 5.2 fail at this frequency and it is thus necessary to return to the original equations of motion for the inviscid fluid. Substituting $\omega = 3$ in Eq. (5.21) we obtain

$$2iv_\rho + 2v_\varphi = \frac{\partial Q}{\partial \rho} , \tag{5.65a}$$

$$2v_\rho - 2iv_\varphi = -\frac{iQ}{\rho} , \tag{5.65b}$$

$$v_z = -\frac{i}{2}\frac{\partial Q}{\partial z} . \tag{5.65c}$$

From the first two equations it follows that

$$v_\rho - iv_\varphi = -\frac{i}{2}\frac{\partial Q}{\partial \rho} ,$$
$$= -\frac{i}{2}\frac{Q}{\rho} ,$$

so that

$$Q = \rho f(z) \tag{5.66}$$

where f is an arbitrary function of z.

In terms of f,

$$v_z = -\frac{i\rho}{2} f'(z) \tag{5.67}$$

and

$$v_\rho - iv_\varphi = -\frac{i}{2} f(z) . \tag{5.68}$$

To find v_ρ and v_φ, we use the equation of continuity,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{i v_\varphi}{\rho} + \frac{\partial v_z}{\partial z} = 0 \quad ,$$

which can be written as

$$\frac{\partial v_\rho}{\partial\rho} + \frac{2v_\rho}{\rho} = \frac{i}{2}\left[\rho f''(z) - \frac{1}{\rho} f(z)\right] \quad . \tag{5.69}$$

The solution regular at $\rho = 0$ is

$$v_\rho = \frac{i}{8}\left[\rho^2 f''(z) + 2f(z)\right] \quad . \tag{5.71}$$

Finally, to find $f(z)$ we use the condition

$$\rho v_\rho + z v_z = v_r(\infty)$$

at $r = 1$. In terms of $f(z)$, and taking into account that, at $r = 1$, ρ can be replaced by $(1-z^2)^{\frac{1}{2}}$, this condition becomes

$$(1-z^2)f''(z) - 4zf'(z) - 2f(z) = -8i(1-z^2)^{-\frac{1}{2}} v_r(\infty) \quad .$$

This equation can also be written in the form

$$\frac{d^2}{dz^2}\left[(1-z^2)f(z)\right] = -8i(1-z^2)^{-\frac{1}{2}} v_r(\infty) \quad , \tag{5.72}$$

which makes the integration very simple.

The variable $v_r(\infty)$ can be obtained by setting $\omega = 3$ in Eq. (5.17). The result is

$$v_r(\infty) = \frac{3}{8}(1-i)\epsilon\left[(1-\cos\theta)^{\frac{1}{2}} - (1+\cos\theta)^{\frac{1}{2}}\right] \quad ,$$

$$= \frac{3}{8}(1-i)\epsilon\left[(1-z)^{\frac{1}{2}} - (1+z)^{\frac{1}{2}}\right] \quad . \tag{5.73}$$

The two integrations required to find $f(z)$ are easily carried out, and it is possible to choose the constants of integration in such a way as to make $f(z)$ finite at $z = \pm 1$. The final form of $f(z)$ is

$$f(z) = \frac{4\epsilon(1+i)}{1-z^2} \left[(1-z)^{3/2} - (1+z)^{3/2} + 2^{3/2} z \right] \quad . \tag{5.74}$$

The velocity components can then be obtained through Eqs. (5.67), (5.70) and (5.71).

It is interesting to note that the interior flow is proportional to ϵ. The singularities in the velocity components at $z = \pm 1$ are only apparent since at those values of z, $\rho = 0$, and the derivatives of $f(z)$ appear multiplied by powers of ρ.

6. DISCUSSION

6.1 Summary of the Results Obtained

In this work we have discussed the problem of small angle precession both for very high and very low viscosity. Physically, the case of high viscosity is characterized by rapid adjustment of the fluid to the instantaneous axis of rotation so that the motion is essentially that of a rigid body. In Chapter 4 the problem for a highly viscous fluid is treated by a series expansion in inverse powers of the viscosity. This approach appears to be very successful, and in principle could be carried to any desired accuracy. The answer obtained is not a singular perturbation but rather a legitimate series expansion so that it could presumably be used for the case of low viscosity as well, except that it would then be hindered by slow convergence. The increasing difficulty in obtaining successive terms in the expansion would make it impractical to use that expansion except for very large values of the viscosity.

The physical situation is entirely different in the case of low viscosity. The division of the flow into slowly precessing and rapidly precessing does not depend on the viscosity; it is merely a matter of the relative magnitudes of the precession and rotation speeds. Yet the viscosity is still important in determining the flow, since, if the viscosity were exactly zero, the fluid could be undergoing any motion consistent with the equations of motion and the boundary condition on the radial flow. The fluid motion would not have to have the same time dependence as the boundary and would in fact be quite independent of the motion of the boundary.

The viscosity affects the interior flow by an exchange of fluid between the boundary layer and the interior. In the discussion of the boundary layer it was found that $v_r(\infty)$ is not zero so that the boundary layer draws fluid from the interior flow at some points and returns fluid to it at others. In this way the boundary layer provides the driving force that moves the interior. In the case of the resonant modes described by Greenspan, the boundary layer drives the interior motion in such a way as to produce resonance.

The role of the critical circles is not too clear. The results obtained indicate that at these circles there is an enhanced exchange of fluid between the boundary layer and the interior since $v_r(\infty)$ is much larger there. Our success in dealing with this part of the problem was limited. The method which was used to find an approximate solution to the flow near the circles has certain interesting features. It is reminiscent of the W.K.B. approximation used in wave propagation problems. The relationship that seems to exist between the approximation and the asymptotic expansion of a contour integral is intriguing, and again there are similarities with the connection formulas used to join two W.K.B. solutions across a turning point. Further work on this part of the problem is required; an exact solution to the modified boundary layer equations would be extremely valuable. It is needed to clarify the physical role of the critical circles and to explain their presence.

6.2 Application to the Bondi-Lyttleton Problem

The problem we have discussed differs from the precession problem posed by Bondi and Lyttleton in that we have taken the precession angle to be small, while those authors linearized the problem with respect to the precession frequency. In Appendix B the equations of motion appropriate to the Bondi-Lyttleton problem are derived, and it is shown that this problem is a more adequate model of the precessional effect on the internal motions of the earth's core, since of the two precessional motions that the earth undergoes it is the one of its axis of rotation about the normal to the ecliptic that is more important.

Comparing Eqs. (B.11) and (B.12) with the equations obtained by setting $\omega = 0$ in Eqs. (3.19) and (3.20), we see that the two problems are essentially the same. The equations in Appendix B are inhomogeneous, but the velocity is subject to homogeneous boundary conditions. It is possible to make the equations homogeneous by substituting

$$\begin{aligned} v_r &= ir \sin\theta \cos\theta + v'_r \quad , \\ v_\theta &= -ir \sin^2\theta + v'_\theta \quad , \\ v_\varphi &= v'_\varphi \quad ; \end{aligned} \tag{6.1}$$

in which case $\vec{v}'$ has to satisfy the boundary conditions

$$\left.\begin{aligned} v'_r &= -i \sin\theta \cos\theta \quad , \\ v'_\theta &= i \sin^2\theta \quad , \\ v'_\varphi &= 0 \quad , \end{aligned}\right\} \quad \text{at } r = 1 \quad . \tag{6.2}$$

Thus, the two problems differ only in the boundary conditions.

The expansion for high viscosity can be carried out in the same manner as before. Only the first correction to rigid body motion was calculated, with the results:

$$v_r = 0 \quad ,$$

$$v_\theta = -\frac{1}{20\epsilon^2}(r^3 - r) \quad , \tag{6.3}$$

$$v_\varphi = \frac{i}{20\epsilon^2}(r^3 - r)\cos\theta \quad .$$

This correction represents a purely toroidal velocity field.

The case of low viscosity presents two kinds of difficulties. The boundary layer solution contains the critical circles, but these can be dealt with in the manner described in Chapter 5. The need for a more precise treatment of the critical circles is again evident. The second difficulty arises in the inviscid problem and was not present in the case of small angle precession. If we consider, as Bondi and Lyttleton did, the inviscid equations for $\vec{v}'$, it can be shown that no solution exists for Q, regular throughout the interior of the sphere and satisfying the condition given in Eq. 6.2

$$v_r' = -i \sin\theta \cos\theta \quad ,$$

$$= -\frac{1}{3} i P_2^1(\cos\theta) \quad , \tag{6.4}$$

at $r = 1$. The proof given by Bondi and Lyttleton is rather involved, but it is possible to use the methods discussed in Section 5.6 to demonstrate this theorem in a relatively simple manner.

It is desired to find a Q such that at the boundary v_r is given by Eq. (6.4). We are dealing with the case $\omega = 0$, or $\cos\beta = -\frac{1}{2}$.

Using the results of Section 5.6, and assuming Q is regular inside the sphere, we can set

$$Q = AP_2^1(\xi)P_2^1(\eta) \quad , \tag{6.4}$$

the boundary being given by $\xi = -\frac{1}{2}$. The condition on Q is

$$A\left[\frac{dP_2^1}{dx} + \frac{P_2^1}{1-x^2}\right]_{x=-\frac{1}{2}} = -\frac{1}{3}i \quad .$$

The term in brackets is zero, so there is no solution of the type sought. The purely inviscid problem can only be solved by admitting solutions of the type

$$Q = [AP_2^1(\xi) + BQ_2^1(\xi)]P_2^1(\eta) \quad , \tag{6.5}$$

which are singular on the line $\rho=0$, or discontinuous solutions which take the form (6.4) in some regions and the form (6.5) in others. Whether such a solution would satisfy all the other physical requirements has not been determined. Another possibility is that a radically different type of boundary layer solution may be needed, which makes $v_r'(\infty)$ equal to zero. Both approaches warrant further investigation, and it is our intention to pursue this subject further in subsequent work.

APPENDIX A

A.1 Iteration in Inverse Powers of the Viscosity

In Section 4.1 the procedure for carrying out the iterations for the expansion of $\vec{v}$ and Q in powers of ϵ^{-2} is discussed. It consists of repeated application of a sequence of formulas which yield T_n, S_n, and Q_n from a knowledge of T_{n-1}, S_{n-1}, and Q_{n-1}. The process starts with a knowledge of T_o and S_o:

$$T_o = r \sin\theta \quad , \tag{A.1}$$

$$S_o = 0 \quad . \tag{A.2}$$

The first step is to find $\widetilde{Q}_o$ by making use of Eq. (4.14):

$$\nabla^2 Q_o = \frac{2}{r} \cos\theta \, L^2 T_o - \sin\theta \frac{\partial^2}{\partial r \partial\theta} (r T_o) + 2iS_o \quad ,$$

$$= \frac{2}{r} \left[\cos\theta (2r \sin\theta) - \sin\theta (2r \cos\theta)\right] + 0 \quad ,$$

$$= 0 \quad .$$

A particular solution of this equation is, of course,

$$\widetilde{Q}_o = 0 \quad . \tag{A.3}$$

The components of $\vec{v}_o$ are found from T_o and S_o by means of Eq. (4.11):

$$\left.\begin{aligned} v_{or} &= 0 \quad , \\ v_{o\theta} &= i r \quad , \\ v_{o\varphi} &= -r \cos\theta \quad . \end{aligned}\right\} \tag{A.4}$$

This information enables us to evaluate the right hand side of Eq. (4.18) and hence to find s_o. As an intermediate step to this end, we evaluate the components of $A\vec{v}_o$:

$$\left.\begin{aligned}(A\vec{v}_o)_r &= 2r\cos\theta\sin\theta \quad ,\\ (A\vec{v}_o)_\theta &= 2r\cos^2\theta \quad ,\\ (A\vec{v}_o)_\varphi &= 2ir\cos\theta \quad .\end{aligned}\right\} \tag{A.5}$$

Equation (4.18) then becomes

$$\begin{aligned}L^2 s_o &= r(A\vec{v}_o)_r + r\,\frac{\partial\widetilde{Q}_o}{\partial r} \quad ,\\ &= 2r^2\cos\theta\sin\theta \quad ,\\ &= \frac{2}{3}\,r^2 P_2^1(\cos\theta) \quad .\end{aligned}$$

Since

$$L^2 P_\ell^1(\cos\theta) = \ell(\ell+1)P_\ell^1(\cos\theta) \quad ,$$

this equation is easily inverted into

$$s_o = \frac{1}{9}\,r^2\,P_2^1(\cos\theta) \quad . \tag{A.6}$$

All the information is now available to obtain t_o from Eq. (4.19)

$$\begin{aligned}t_o &= -i\sin\theta\left[(A\vec{v}_o)_\theta + \frac{1}{r}\,\frac{\partial\widetilde{Q}_o}{\partial\theta} - \frac{1}{r}\,\frac{\partial^2}{\partial r\partial\theta}\,(rs_o)\right] \quad ,\\ &= -i\sin\theta[2r\cos^2\theta - r\cos 2\theta] \quad , \qquad\qquad (A.7)\\ &= -ir\sin\theta \quad .\end{aligned}$$

T_1 can now be obtained from Eq. (4.20), which is

$$\nabla^2 T_1 = -i(\omega-1)T_o + t_o \quad ,$$
$$= -i(\omega-1)r \sin\theta - ir\sin\theta \quad ,$$
$$= -i\omega r \sin\theta \quad . \tag{A.8}$$

This equation must be solved subject to the boundary condition $T_1 = 0$ at $r = 1$ as given in Eq. (4.13).

Equation (A.8) is of the general form

$$\nabla^2 F = r^n P^1_\ell(\cos\theta) \quad . \tag{A.9}$$

Since all the equations encountered in the iteration that follow from Eqs. (4.14), (4.20) and (4.21) are of this form, we digress momentarily to obtain the solution of this equation. Writing F in the form

$$F = f(r)\, P^1_\ell(\cos\theta) \quad ,$$

Eq. (A.9) becomes

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{\ell(\ell+1)f}{r^2} = r^n \quad . \tag{A.10}$$

It is clear that a particular solution is $c_n r^{n+2}$, where c_n must be evaluated by direct substitution into Eq. (A.10):

$$c_n[(n+2)(n+1) + 2(n+2) - \ell(\ell+1)] = 1 \quad ,$$

or,

$$c_n = \frac{1}{(n+2)(n+3) - \ell(\ell+1)} \quad .$$

In addition, the homogeneous solution to (A.10) is cr^ℓ (we discard the singular solution $r^{-(\ell+1)}$), so that

$$F = \left[\frac{r^{n+2}}{(n+2)(n+3) - \ell(\ell+1)} + cr^{\ell} \right] P_{\ell}^{1}(\cos\theta) \quad . \quad (A.11)$$

Applying this to Eq. (A.9), we immediately obtain

$$T_1 = -\frac{i\omega r^3}{10} \sin\theta + cr\sin\theta \quad ,$$

and making use of the condition that $T_1 = 0$ at $r = 1$, we get

$$T_1 = -\frac{i\omega r}{10}(r^2 - 1)\sin\theta \quad . \qquad (A.12)$$

The next step is to find χ_o by using Eqs. (4.23) and (4.24). The term appearing in the integral in the numerator of (4.24) is

$$i(\omega-1)S_o - s_o = -\frac{1}{9} r^2 P_2^1(\cos\theta) \quad ,$$

so that, because of the orthogonality of the associated Legendre polynomials, the only term in the expansion of χ_o is $a_2 r^2 P_2^1(\cos\theta)$. Using Eq. (4.24) we find a_2 to be

$$a_2 = \frac{\iint -\frac{1}{9} r^2 P_2^1(\cos\theta)\,(r^2 P_2^1(\cos\theta))\sin\theta\, d\theta\, r^2 dr}{\iint (r^2 P_2^1(\cos\theta))^2 \sin\theta\, d\theta\, r^2 dr} \quad ,$$

$$= -\frac{1}{9} \quad ,$$

so that

$$\chi_o = -\frac{1}{9} r^2 P_2^1(\cos\theta) \quad , \qquad (A.13)$$

and it then follows from Eq. (4.15) that

$$Q_o = -\frac{1}{3} r^2 P_2^1(\cos\theta) \quad . \qquad (A.14)$$

The first iteration is completed with the evaluation of S_1, which according to Eq. (4.21) is given by

$$\nabla^2 S_1 = -i(\omega-1)S_o + s_o + \chi_o \quad ,$$

$$= 0 + \frac{1}{9}\ r^2 P_2^1 (\cos\theta) - \frac{1}{9}\ r^2 P_2^1(\cos\theta) \quad ,$$

$$= 0 \quad .$$

S_1 is subject to the condition $S_1 = 0$ at $r = 1$, so that the solution is

$$S_1 = 0 \quad . \tag{A.15}$$

Subsequent iterations are found by following exactly the same procedure. The steps become more involved because more harmonics appear at the various stages of the integration. Without going through the details of the next iteration, the results are listed below:

$$Q_1 = \frac{i\omega r^2}{105}\ (3r^2 - 7)\sin\theta\cos\theta \quad , \tag{A.17}$$

$$T_2 = \frac{\omega^2 r}{1400}\ (5r^2 - 9)(r^2 - 1)\sin\theta \quad , \tag{A.18}$$

$$S_2 - \frac{i\omega r^2}{2520}\ (r^2 - 1)^2 \sin\theta\cos\theta \quad . \tag{A.19}$$

To the accuracy of this approximation the velocity components are found to be:

$$v_r = -\frac{1}{\epsilon^4}\ \frac{i\omega r}{420}\ (r^2 - 1)^2 \sin\theta\cos\theta \quad , \tag{A.20a}$$

$$v_\theta = ir + \frac{1}{\epsilon^2}\ \frac{\omega r}{10}\ (r^2 - 1)$$

$$- \frac{1}{\epsilon^4}\ \frac{i\omega r}{2520}\ (7r^2 - 3)\cos 2\theta + \frac{i\omega^2 r}{1400}\ (5r^2 - 9)\ (r^2 - 1) \quad . \tag{A.20b}$$

and

$$v_\varphi = -r\cos\theta + \frac{1}{\epsilon^2}\frac{i\omega r}{10}(r^2-1)\cos\theta + \frac{1}{\epsilon^4}\frac{\omega r}{2520}(7r^2-3) + \frac{\omega^2 r}{1400}(5r^2-9)(r^2-1)\cos\theta \quad . \quad (A.20c)$$

The complex pressure Q, to order ϵ^{-2} is

$$Q = -r^2\sin\theta\cos\theta + \frac{1}{\epsilon^2}\frac{i\omega r^2}{105}(3r^2-7)\sin\theta\cos\theta \quad . \qquad (A.21)$$

APPENDIX B

B.1 Equations of Motion for Slow Precession

The equations of motion for the case of slow precession $(\vec{\Omega} \to 0)$ are most easily derived in the precessing frame, that is in a frame of reference which is rotating with an angular velocity $\vec{\Omega}$. This is the frame that was used by Bondi and Lyttleton[22], although their derivation contained several mistakes which were later corrected by Stewartson and Roberts[27]. The errors that appear in Ref. 22 were such that the final equations were substantially correct and the results obtained by Bondi and Lyttleton were unaffected by these errors.

The momentum equation for an incompressible fluid in a rotating reference frame is

$$\frac{\partial \vec{q}}{\partial t} - \vec{q} \times (\nabla \times \vec{q}) + 2\Omega \times \vec{q} - \nu \nabla^2 \vec{q} = -\nabla(p/\rho + \frac{1}{2} q^2) \quad , \tag{B.1}$$

where $\vec{q}$ and p are the velocity and pressure relative to the rotating frame, and $\vec{\Omega}$ is its angular velocity. The continuity equation remains unchanged,

$$\nabla \cdot \vec{q} = 0 \quad . \tag{B.2}$$

In the rotating reference frame, the angular velocity of the shell, $\vec{\omega}_r$ is transformed into a constant vector $\vec{\omega}_s$, where

$$\vec{\omega}_s = \vec{\omega}_R - \vec{\Omega} \quad . \tag{B.3}$$

The motion of the boundary is thus described by

$$\vec{q} = \vec{\omega}_s \times r \quad ,$$

and following Bondi and Lyttleton, we set

$$\vec{q} = \vec{\omega}_s \times \vec{r} + \vec{u} \tag{B.4}$$

and substitute into Eq. (B.1). The procedure parallels that described in Section 3.1, since Eq. (B.1) differs from Eq. (3.2) only by the presence of the term $2\vec{\Omega} \times \vec{q}$ in the left hand side of Eq. (B.1). In terms of the velocity relative to the boundary, $\vec{u}$, this term is

$$2\vec{\Omega} \times (\vec{\omega}_s \times \vec{r}) + 2\vec{\Omega} \times \vec{u} \quad ,$$

so that, borrowing the results of Section 3.1 we may write

$$\frac{\partial \vec{u}}{\partial t} - (\vec{\omega}_s \times \vec{r}) \times (\nabla \times \vec{u}) + 2\vec{\omega}_s \times \vec{u} - \vec{u} \times (\nabla \times \vec{u})$$
$$+ 2\vec{\Omega} \times \vec{u} + 2\vec{\Omega} \times (\vec{\omega}_s \times \vec{r}) - \nu \nabla^2 \vec{u}$$
$$= - \nabla \left[p/\rho + \frac{1}{2} u^2 - \frac{1}{2} (\vec{\omega}_s \times \vec{r})^2 + \vec{u} \cdot (\vec{\omega}_s \times \vec{r}) \right] \quad . \qquad (B.5)$$

Since $\vec{u} = 0$ is a solution to the equations of motion when $\vec{\Omega} = 0$, it can be assumed that for sufficiently small values of Ω, $\vec{u}$ is a linear function of $\vec{\Omega}$. Equation (B.5) can thus be linearized to first order terms in $\vec{\Omega}$:

$$\frac{\partial \vec{u}}{\partial t} - (\vec{\omega}_s \times \vec{r}) \times (\nabla \times \vec{u}) + 2\vec{\omega}_s \times \vec{u} - \nu \nabla^2 \vec{u}$$
$$= -2\vec{\Omega} \times (\vec{\omega}_s \times \vec{r}) - \nabla \left[p'/\rho + \vec{u} \cdot (\vec{\omega}_s \times \vec{r}) \right] \quad ,$$

where

$$p' = p - \frac{1}{2} \rho (\vec{\omega}_s \times \vec{r})^2 \quad .$$

It is convenient to split the Coriolis term in the following way

$$\vec{\Omega} \times (\vec{\omega}_s \times r) = \vec{\omega}_s \left[\vec{\Omega} \cdot \vec{r} - \frac{(\vec{\omega}_s \cdot \vec{\Omega})(\vec{\omega}_s \cdot \vec{r})}{\omega_s^2} \right] - \frac{1}{2} \frac{(\vec{\omega}_s \cdot \vec{\Omega})}{\omega_s^2} \nabla (\vec{\omega}_s \times \vec{r})^2 \quad ,$$

so that the linearized equation finally becomes

$$\frac{\partial \vec{u}}{\partial t} + \nabla[\vec{u}\cdot(\vec{\omega}_s \times \vec{r})] - (\vec{\omega}_s \times \vec{r}) \times (\nabla \times \vec{u}) + 2\vec{\omega}_s \times \vec{u}$$

$$- \nu \nabla^2 \vec{u} = \vec{a} - \omega_s \nabla P \quad , \qquad \text{(B.6)}$$

where

$$\vec{a} = 2\vec{\omega}_s \left[\frac{(\vec{\omega}_s \cdot \vec{\Omega})(\vec{\omega}_s \cdot \vec{r})}{\omega_s^2} - \vec{\Omega} \cdot \vec{r} \right] ,$$

and

$$\omega_s P = p'/\rho - \frac{\vec{\omega}_s \cdot \vec{\Omega}}{\omega_s^2} (\vec{\omega}_s \times \vec{r})^2 \quad .$$

Except for the forcing term $\vec{a}$, Eq. (B.6) is the same as Eq. (3.8) with $\vec{\omega}_o$ replaced by $\vec{\omega}_s$. It is therefore convenient to take the z-axis of the coordinate system along $\vec{\omega}_s$. We shall also assume $\vec{\Omega}$ lies in the x-z plane so that

$$\vec{\Omega} = \Omega \sin\alpha\, \vec{e}_x + \Omega \cos\alpha\, \vec{e}_z \quad ,$$

in which case $\vec{a}$ is given by

$$\vec{a} = -2\omega_s \Omega \sin\alpha\; x \vec{e}_z \quad . \qquad \text{(B.7)}$$

It should be noted that since $\vec{a}$ is the driving term in the equations of motion, it is $\Omega \sin\alpha$, rather than Ω, which determines the fluid velocity. In the case of the earth, the forced precession has a period of 26,000 years and the angle α is $23.4°$, while the free precession has a period of 300 days with an angle α of about 10^{-6} radians, so that $\Omega \sin\alpha$ has a value of 3×10^{-12} $\sec^{-1}$ for the forced precession, and of 3×10^{-14} $\sec^{-1}$ for the free precession. It follows that the

26,000 year precession is more important in this case.

Returning to the equations of motion, the components of the momentum equation for the present problem can be written directly by referring to Eq. (3.13):

$$\frac{1}{\omega_s}\frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial \varphi} - 2\sin\theta\, u_\varphi - \frac{\nu}{\omega_s}(\nabla^2\vec{u})_r$$

$$= -\frac{\partial P}{\partial r} - 2\Omega\sin\alpha\, r\sin\theta\cos\theta\cos\varphi \quad , \qquad \text{(B.8a)}$$

$$\frac{1}{\omega_s}\frac{\partial u_\theta}{\partial t} + \frac{\partial u_\theta}{\partial \varphi} - 2\cos\theta\, u_\varphi - \frac{\nu}{\omega_s}(\nabla^2\vec{u})_\theta$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + 2\Omega\sin\alpha\, r\sin^2\theta\cos\varphi \quad , \qquad \text{(B.8b)}$$

$$\frac{1}{\omega_s}\frac{\partial u_\varphi}{\partial t} + \frac{\partial u_\varphi}{\partial \varphi} + 2(\cos\theta\, u_\theta + \sin\theta\, u_r) - \frac{\nu}{\omega_s}(\nabla^2\vec{u})_\varphi$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \varphi} \quad . \qquad \text{(B.8c)}$$

Upon substituting

$$\vec{u} = \mathrm{Re}\{2\Omega\, a\, \sin\alpha\, \vec{v}(r,\theta)e^{i\varphi}\} \quad , \qquad \text{(B.9)}$$

$$P = \mathrm{Re}\{2\Omega\, a^2 \sin\alpha\, Q(r,\theta)e^{i\varphi}\} \quad , \qquad \text{(B.10)}$$

into Eq. (B.8), and expressing all lengths as a fraction of the radius, we obtain

$$iv_r - 2\sin\theta\, v_\varphi - \epsilon^2(\nabla^2\vec{v})_r = -\frac{\partial Q}{\partial r} - r\sin\theta\cos\theta \quad , \qquad \text{(B.11a)}$$

$$iv_\theta - 2\cos\theta\, v_\varphi - \epsilon^2(\nabla^2\vec{v})_\theta = -\frac{1}{r}\frac{\partial Q}{\partial \theta} + r\sin^2\theta \quad , \qquad \text{(B.11b)}$$

$$iv_\varphi + 2(\cos\theta\, v_\theta + \sin\theta\, v_r) - \epsilon^2(\nabla^2\vec{v})_\varphi = -\frac{iQ}{r\sin\theta} \quad , \qquad \text{(B.11c)}$$

where $\epsilon^2 = \nu/a^2\omega_s$. The complex velocity $\vec{v}$ must also satisfy the equation of continuity

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{i v_\varphi}{r\sin\theta} = 0 \quad , \tag{B.12}$$

and is subject to the condition $\vec{v} = 0$ at the boundary.

II. ELECTROMAGNETIC RADIATION FROM AN EXPANDING SPHERE IN A MAGNETIC FIELD

1. INTRODUCTION

The work presented in the pages that follow arose out of an attempt to clarify certain aspects of the interaction between an electromagnetic field and a material object moving in it. In particular, it was desired to study in detail the role of the field energy in such a situation. It was felt that an understanding of the energy balance in a process in which a field interacts with a material medium would be useful in formulating magnetohydrodynamic problems from the point of view of energy conservation.

Specifically, the magnetohydrodynamic problem we were interested in was that of the adiabatic expansion of a cloud of conducting gas in a uniform magnetic field. This problem proved to be of extreme complexity, and on exploring possible approaches that would simplify the problem, we attempted to formulate it in terms of overall energy conservation. It was hoped that some information about the shape of the expanding cloud would be obtained by using the condition that the sum of the kinetic and internal energies of the gas and the energy in the field has to be constant. In setting up the problem in this way we found that a distinction must be made between the energy stored in the field that is available as work and what is ordinarily called the field energy.

To understand this distinction fully we undertook to study in detail the following problem: a perfectly conducting sphere is placed in a uniform magnetic field and by some external means the radius of the sphere is changed. An energy balance was then made which showed what happens to the energy between the time the original static field is disturbed and the time the transients have disappeared and the field is again static. Some periodic motions of the radius were also considered. The results of this study were reported in a paper by the author[1], and are given here with minor changes. In a second paper by M. S. Plesset and the author[2] the torque on a permeable ellipsoid in a uniform magnetic field was calculated by energy methods. In Ref. 2 only the initial and final states were considered without a detailed consideration of the flow of energy in the interval of time during which the field changes from its initial to its final configuration. The results obtained in that paper apply to the interaction of a uniform field with a material medium of finite extent but of arbitrary geometry. Because of the more general scope of this problem, it is impossible to analyze the transients. Moreover, only slow changes are considered, in contrast with the problem studied in Ref. 1, where the effects of the speed of the boundary are fully taken into account. Only the work that appears in Ref. 1 is reported here.

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[1] G. Venezian, "Radiation due to the Radial Motion of a Conducting Sphere in a Magnetic Field," Report No. 85-25, Div. of Eng. and App. Sci., California Institute of Technology (June, 1963)

[2] M. S. Plesset and G. Venezian, "Free Energy in Magnetostatic or Electrostatic Fields," Am. J. Phys. 32, 860 (1965).

2. FORMULATION OF THE PROBLEM

2.1 Field Equations

The problem we are considering is to find the electromagnetic fields outside a spherical region of varying radius. It is assumed that in this region there are no sources, so that the fields obey Maxwell's equations for a vacuum[3],

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}, \qquad (2.1)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}, \qquad (2.2)$$

$$\nabla \cdot \vec{B} = 0 , \qquad (2.3)$$

$$\nabla \cdot \vec{D} = 0 , \qquad (2.4)$$

where

$$\vec{D} = \epsilon_o \vec{E} \quad \text{and} \quad \vec{B} = \mu_o \vec{H} .$$

(M. K. S. units will be used throughout.)

Since there are no charges, it is possible to derive these fields from a vector potential $\vec{A}$ by putting

$$\vec{B} = \nabla \times \vec{A} , \qquad (2.5)$$

then by Eq. (2.1)

$$\vec{E} = - \frac{\partial \vec{A}}{\partial t} \qquad (2.6)$$

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[3] W. Panofsky and M. Phillips, Classical Electricity and Magnetism (Addison-Wesley, Reading, Pa. 1956), p. 143.

and from Eq. (2.4) we obtain that

$$\frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = 0 ,$$

which implies that $\nabla \cdot \vec{A}$ is a function of the space coordinates only. A function can be incorporated into $\vec{A}$ in such a way that

$$\nabla \cdot \vec{A} = 0 . \tag{2.7}$$

By putting Eqs. (2.5) and (2.6) into (2.2) we obtain the wave equation

$$\nabla \times (\nabla \times \vec{A}) + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0 . \tag{2.8}$$

Equation (2.7) shows that only two components of $\vec{A}$ are independent. To solve a problem with cylindrical symmetry, it is convenient to write $\vec{A}$ in terms of two scalar functions u and v, instead of in component form, as follows:

$$\vec{A}(r,\theta,t) = \vec{e}_\varphi u(r,\theta,t) + \nabla \times \vec{e}_\varphi v(r,\theta,t) \tag{2.9}$$

where (r,θ,φ) are the spherical polar coordinates of the field point and $\vec{e}_\varphi$ is the unit vector in the φ direction. Here we have explicitly shown that $\vec{A}$ is not a function of the angle φ, and is thus cylindrically symmetric. When $\vec{A}$ is written in this form, $\nabla \cdot \vec{A}$ is zero; the divergence of the first term is zero because it does not depend on φ, and the second term has zero divergence because it is the curl of a vector.

From Eq. (2.9)

$$\nabla \times \vec{A} = \vec{e}_r \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta u) - \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial r} (ru)$$

$$- \vec{e}_\varphi \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} (rv) + \frac{1}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta v) \right) \right], \quad (2.10)$$

and

$$\nabla \times (\nabla \times \vec{A}) = - \vec{e}_r \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} (rv) + \frac{1}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta v) \right) \right]$$

$$+ \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial r} \left[\frac{\partial^2}{\partial r^2} (rv) + \frac{1}{r} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta v) \right) \right]$$

$$- \vec{e}_\varphi \; \frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) + \frac{1}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta u) \right) . \quad (2.11)$$

It follows that both u and v must satisfy

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) + \frac{1}{r^2} \frac{\partial}{\partial\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta u) - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 . \quad (2.12)$$

In Appendix A it is shown that the solution of this equation satisfying the radiation condition of outgoing waves at infinity is

$$u = \sum \psi_n(r,t) P_n^1(\cos\theta), \quad (2.13)$$

where

$$\psi_n(r,t) = r^n \left[\frac{1}{r} \frac{\partial}{\partial r} \right]^n \frac{f_n(t - r/c)}{r}, \quad (2.14)$$

and $f_n(t - r/c)$ is an arbitrary function, which must be determined from the boundary and initial conditions of the problem.

The components of the field vectors can be expressed in terms of u and v by making use of Eqs. (2.5), (2.6), (2.10) and (2.11). The resulting expressions are:

$$B_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta u) ,$$

$$B_\theta = - \frac{1}{r} \frac{\partial}{\partial r} (ru) ,$$

$$B_\varphi = - \left[\frac{1}{r} \frac{\partial^2}{\partial r^2} (rv) + \frac{1}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta v) \right) \right] ,$$

$$E_r = - \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial v}{\partial t} \right) ,$$

$$E_\theta = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial t} \right) .$$

We shall need these formulas only for the special case $v = 0$, $u = F(r,t) \sin\theta$. In this case the field components become

$$B_r = \frac{2}{r} F \cos\theta , \tag{2.15}$$

$$B_\theta = - \frac{1}{r} \frac{\partial(rF)}{\partial r} \sin\theta , \tag{2.16}$$

$$B_\varphi = E_r = E_\theta = 0 , \tag{2.17}$$

$$E_\varphi = - \frac{\partial F}{\partial t} \sin\theta . \tag{2.18}$$

Also, since $\sin\theta = P_1^1(\cos\theta)$, from Eqs. (2.14) and (2.15) it follows that $F(r,t)$ must be of the form

$$F(r,t) = -\frac{f(t - r/c) + \frac{r}{c} f'(t - r/c)}{r^2} \quad . \tag{2.19}$$

Note that a term Kr may be added to F since $r \sin\theta$ is also a solution to Eq. (2.12). The expression $\sin\theta/r^2$ is also a solution, but this can be included in f by adding a constant to it.

2.2 Boundary Conditions

To solve Maxwell's equations we must know the conditions that the field vectors satisfy at physical boundaries and at infinity. At a stationary interface between two media the field vectors satisfy the conditions

$$\vec{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

$$\vec{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0$$

where $\vec{n}$ is the normal to the boundary. If medium 2 is a perfect conductor, $\vec{E}_2 = 0$ and $\nabla \times \vec{E}_2 = 0$. Therefore, by Eq. (1.1) the time rate of change of $\vec{B}_2$ is zero, and if we assume $\vec{B}_2$ was equal to zero initially, the boundary conditions for a stationary conducting surface become

$$\vec{n} \times \vec{E} = 0$$

$$\vec{n} \cdot \vec{B} = 0 \, . \tag{2.20}$$

For time varying fields, only one of these conditions is independent, since $\vec{E}$ and $\vec{B}$ are connected through Maxwell's equations.

The boundary conditions are changed if the surface is moving. For a boundary moving with velocity $\vec{v}$, the correct conditions are[4]

$$\vec{n} \times (\vec{E} + \vec{v} \times \vec{B}) = 0$$

$$\vec{n} \cdot \vec{B} = 0 \, . \tag{2.21}$$

Again, only one of these is independent for time varying fields. In our case there will be no static electric fields so that it will be convenient to use the condition $\vec{n} \cdot \vec{B} = 0$ since it applies to both stationary and moving surfaces. We shall see that when this condition is applied, the tangential electric field will be zero if the surface is stationary or $-\vec{n} \times (\vec{v} \times \vec{B})$ if it is moving.

We must also impose the condition that at infinity the magnetic field should reduce to $B_o \vec{e}_z$, with components

$$B_r = B_o \cos \theta \, ,$$

$$B_\theta = - B_o \sin \theta \, . \tag{2.22}$$

The corresponding vector potential has only one component,

$$A_\varphi = \frac{1}{2} B_o r \sin \theta \, . \tag{2.23}$$

[4] R. Tolman, Relativity Thermodynamics and Cosmology (Oxford, 1934), p. 112.

3. UNIFORMLY EXPANDING SPHERE

3.1 Fields

In this section we consider the problem of a sphere that starts from zero radius at $t = 0$ in a uniform field $B_o \vec{e}_z$, and expands at a uniform speed v until it reaches a radius $\underline{a}$. The sphere then stops expanding.

In Fig. 1 the motion of the surface of the sphere is shown in the r-t plane as the line OPN. This line, and the characteristics PQ and OR divide the plane into four regions. In region I the motion of the sphere has not yet been felt, and the field is the uniform field that existed at $t = 0$. In region II the signal generated by the expanding sphere affects the fields. Similarly in region III the fact that the sphere has ceased to expand is felt. Region IV is the interior of the sphere. We must now solve the field equations in regions II and III. From the axial symmetry of the problem it is evident that the magnetic field can have no φ component. Thus v in Eq. (2.9) is zero and the vector potential will only have a φ component. In region I the vector potential is

$$A_\varphi^{(1)} = \frac{1}{2} B_o r \sin\theta \tag{3.1}$$

which gives rise to a uniform field $B_o \vec{e}_z$. In region II, according to Eq. (2.13) we can write

$$A_\varphi^{(2)} = \sum P_n^1(\cos\theta)\psi_n(r,t) .$$

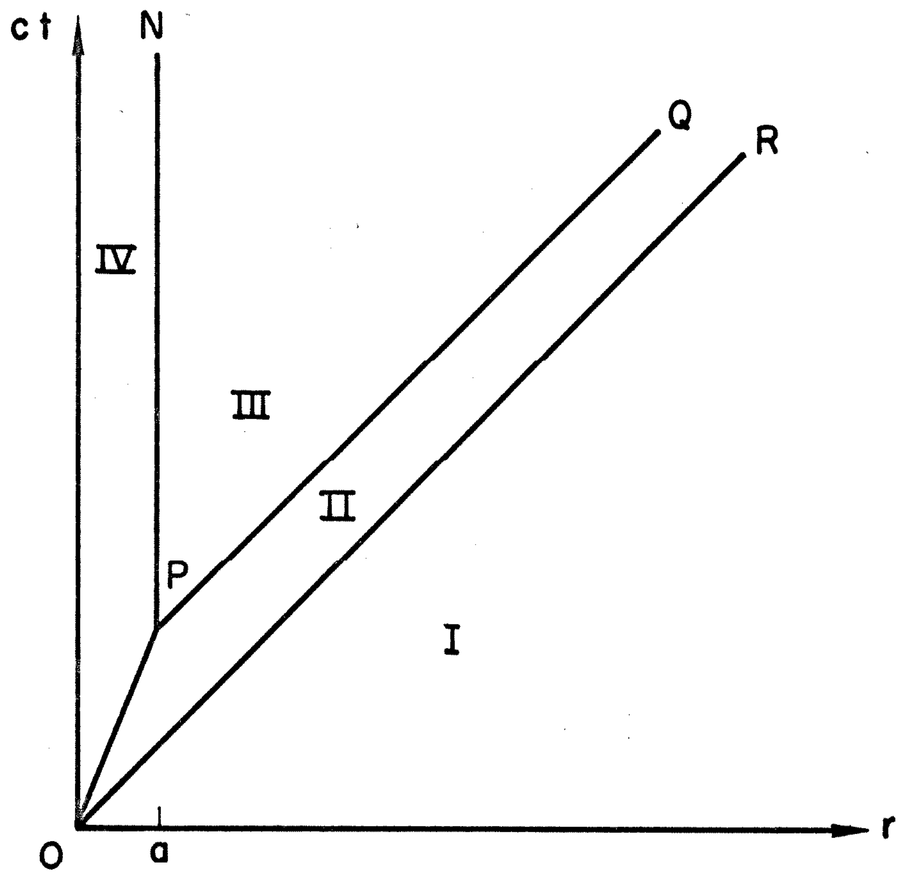

Figure 1. r - ct plane for the expanding sphere.

Since the line $r = ct$, which is the boundary between I and II, is not a physical surface, the field components must be continuous there. It follows that A_φ and its derivatives are continuous. This requires that $A_\varphi^{(1)}$ and $A_\varphi^{(2)}$ have the same dependence on the angle θ. Therefore

$$A_\varphi^{(2)} = K\left[r - \frac{f(\eta) + \frac{r}{c} f'(\eta)}{r^2}\right] \sin\theta, \quad \eta = t - r/c \tag{3.2}$$

as in Eq. (2.19). We have added the term Kr to account for the applied field.

At $r = ct$, $A_\varphi^{(1)} = A_\varphi^{(2)}$, and therefore $K = \frac{1}{2} B_o$ and

$$f(0) = f'(0) = 0 . \tag{3.3}$$

To find $f(\eta)$ we apply the boundary conditions at the surface of the sphere, $r = vt$. On this line $\eta = t(1 - \beta)$ and $r = \beta c\eta/(1 - \beta)$, where $\beta = v/c$. Since $\eta = t(1 - \beta)$, $B_r = 0$ on this line, and using Eq. (3.2) we obtain

$$r - \frac{f(\eta) + \frac{r}{c} f'(\eta)}{r^2} = 0$$

at

$$r = \beta c\eta/(1 - \beta) ,$$

or

$$f'(\eta) + \frac{1 - \beta}{\beta} \frac{1}{\eta} f(\eta) = \frac{c^3\beta^2}{(1 - \beta)^2} \eta^2 . \tag{3.4}$$

This is a linear first order differential equation and can thus be integrated to give

$$f(\eta) = \frac{1}{\eta^{(1-\beta)/\beta}} \int \eta^{(1+\beta)/\beta} \frac{c^3\beta^2}{(1-\beta)^2}\, d\eta \,.$$

Carrying out the integration, and applying the condition $f(0) = 0$ from Eq. (3.3) we obtain

$$f(\eta) = \frac{c^3\beta^3\eta^3}{(1-\beta)^2(1+2\beta)} \tag{3.5}$$

which also satisfies the second condition $f'(0) = 0$. It can be verified that this form of $f(\eta)$ also satisfies the condition $E_\varphi = -vB_\theta$ as required.

The arguments used in obtaining the form of $A_\varphi^{(2)}$ apply to $A_\varphi^{(3)}$. Moreover it is clear that in this case too the coefficient of $r \sin\theta$ is $\frac{1}{2} B_o$. Therefore

$$A_\varphi^{(3)} = \frac{1}{2} B_o \left[r - \frac{g(\eta) + \frac{r}{c} g'(\eta)}{r^2} \right] \sin\theta \,. \tag{3.6}$$

To find $g(\eta)$ we apply the boundary condition $B_r = 0$ at $r = a$. This gives

$$g'(\eta) + \frac{c}{a} g(\eta) = ca^2$$

which can be solved to give

$$g(\eta) = a^3 + M \exp\left[-\frac{c\eta}{a} \right] \,,$$

where M is a constant of integration. This constant can be evaluated by making A_φ continuous across the boundary between regions II and III. The equation of this boundary is $\eta = \frac{a}{c}\,\frac{1-\beta}{\beta}$. Putting $A_\varphi^{(2)} = A_\varphi^{(3)}$ on this line we get the pair of equations

$$a^3 + M \exp\left[-\frac{(1-\beta)}{\beta}\right] = \frac{a^3(1-\beta)}{1+2\beta} ,$$

$$M \exp\left[-\frac{(1-\beta)}{\beta}\right] = -\frac{3a^3\beta}{1+2\beta} ,$$

both of which must hold in order that the previous equation be satisfied for all r on the boundary line. Both expressions lead to the same solution,

$$M = -\frac{3\beta a^3}{1+2\beta} \exp\left[\frac{1-\beta}{\beta}\right]$$

so that

$$g(\eta) = a^3 \left[1 - \frac{3\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a}\right]\right]. \qquad (3.7)$$

We can now write down expressions for the components of $\vec{E}$ and $\vec{B}$ by making use of Eqs. (3.1) - (3.4) and the expressions for $f(\eta)$ and $g(\eta)$ which we have just obtained.

For region II, $\frac{a}{c}\,\frac{1-\beta}{\beta} > \eta > 0$

$$\left.\begin{aligned} B_r &= B_o \cos\theta \left[1 - \frac{K\eta^3 c^3}{r^3} - \frac{3K\eta^2 c^2}{r^2} \right] \\ B_\theta &= -\frac{1}{2} B_o \sin\theta \left[2 + \frac{K\eta^3 c^3}{r^3} + \frac{3K\eta^2 c^2}{r^2} + \frac{6K\eta c}{r} \right] \\ E_\varphi &= \frac{3}{2} K B_o c \sin\theta \left[\frac{\eta^2 c^2}{r^2} + \frac{2\eta c}{r} \right] \end{aligned}\right\} \quad (3.8)$$

where $K = \dfrac{\beta^3}{(1-\beta)^2(1+2\beta)}$ and $\eta = t - r/c$.

In region III,

$$t - \frac{a}{c} > \eta > \frac{a}{c}\,\frac{1-\beta}{\beta}$$

$$B_r = B_o \cos\theta \left[1 - \frac{a^3}{r^3}\left(1 - \frac{3\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] \right) + \frac{3a^2}{r^2}\,\frac{\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] - \frac{3a}{r}\,\frac{\beta}{1+\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] \right]$$

$$B_\theta = -\frac{1}{2} B_o \sin\theta \left[2 + \frac{a^3}{r^3}\left(1 - \frac{3\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] \right) + \frac{3a^2}{r^2}\,\frac{\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] - \frac{3a}{r}\,\frac{\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] \right] \quad (3.9)$$

$$E_\varphi = \frac{3}{2} B_o c \sin\theta \, \frac{\beta}{1+2\beta} \left(\frac{a^2}{r^2} - \frac{a}{r} \right) \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right]$$

The fields evaluated at the surface of the sphere are:

$$B_r = 0$$

$$B_\theta = -\frac{3}{2} B_o \sin\theta \frac{(1+\beta)}{(1-\beta)(1+2\beta)} \tag{3.10}$$

$$E_\varphi = \frac{3}{2} B_o \sin\theta \frac{\beta(1+\beta)}{(1-\beta)(1+2\beta)}$$

for $0 < t < a/v$, and

$$B_r = 0$$

$$B_\theta = -\frac{3}{2} B_o \sin\theta \left(1 - \frac{\beta}{1+2\beta} \exp\left[\frac{1}{\beta} - \frac{ct}{a}\right]\right) \tag{3.11}$$

$$E_\varphi = 0$$

for $t > a/v$. It is interesting to note that while the sphere is expanding the fields at the surface remain constant, and also that the electric field E_φ is not zero but, as required by Eq. (2.21), equal to $-vB_\theta$.

3.2 Energy Balance

It is of interest to determine the amount of work that must be done to make the sphere expand to its final radius, and in what way this energy is expended. To do this we calculate the radiated energy, the energy stored in the field, and the mechanical work required for the expansion. It will be found that energy is actually removed from the field and transformed into radiation. In addition, the work performed against the surface stresses will also appear in the radiation field.

(a) Radiated Energy. To achieve a thorough understanding of the physical principles involved, we evaluate the total flux of energy through the surface of a sphere of radius R, R > a. From Eqs. (3.8) and (3.9) the radial component of the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ is

$$S_r^{(2)} = \frac{3c}{4\mu_o} B_o^2 \sin^2\theta K^2 \left(\frac{\eta^2 c^2}{r^2} + \frac{2\eta c}{r} \right) \left[\frac{2}{K} + \frac{\eta^3 c^3}{r^3} + \frac{3\eta^2 c^2}{r^2} + \frac{6\eta c}{r} \right] , \tag{3.12}$$

$$S_r^{(3)} = -\frac{3c}{4\mu_o} B_o^2 \sin^2\theta \frac{\beta}{1+2\beta} \frac{a}{r} \left(1 - \frac{a}{r} \right) \times$$
$$\left[\left(2 + \frac{a^3}{r^3} \right) \exp\left[\frac{1-\beta}{\beta} - \frac{c\eta}{a} \right] - \frac{3\beta}{1+2\beta} \frac{a}{r} \left(1 - \frac{a}{r} + \frac{a^2}{r^2} \right) \exp\left[\frac{2(1-\beta)}{\beta} - \frac{2c\eta}{a} \right] \right] . \tag{3.13}$$

The total flux of energy through the surface of a sphere of radius R is

$$W_f = \int_o^\infty dt \int_o^{2\pi} d\varphi \int_o^\pi R^2 \sin\theta [S_r]_{r=R} \, d\theta . \tag{3.14}$$

Since S_r is given by the two expressions (3.12) and (3.13), W_f splits into two integrals, $W_f^{(2)}$ and $W_f^{(3)}$, which we now evaluate. In both cases the angular dependence of S_r is $\sin^2\theta$, and thus the integration over angles gives

$$\int_o^{2\pi} d\varphi \int_o^\pi \sin^3 d\theta = \frac{8\pi}{3} . \tag{3.15}$$

A further simplification results if we change the variable of integration from t to $\eta = t - R/c$. With this change,

$$W_f^{(2)} = \frac{2\pi B_o^2 K^2 c R^2}{\mu_o} \int_o^{\frac{a}{c}\frac{1-\beta}{\beta}} \Phi_2(\eta)\, d\eta \ ,$$

where

$$\Phi_2(\eta) = \frac{\eta^5 c^5}{R^5} + \frac{5\eta^4 c^4}{R^4} + \frac{12\eta^3 c^3}{R^3} + \left[12 + \frac{2(1-\beta)^2(1+2\beta)}{B^3} \right] \frac{\eta^2 c^2}{R^2}$$

$$+ \frac{4(1-\beta)^2(1+2\beta)}{\beta^3} \frac{\eta c}{R} \ ,$$

so that

$$W_f^{(2)} = \frac{2\pi B_o^2}{\mu_o} \frac{a^2 R}{(1-\beta)(1+2\beta)^2} \left[\frac{1}{6} \alpha^4 (1-\beta)^3 + \alpha^3 \beta (1-\beta)^2 + 3\alpha^2 \beta^2 (1-\beta) \right.$$

$$+ \alpha \ 4\beta^3 + \frac{2}{3}(1-\beta)^2(1+2\beta)$$

$$\left. + 2\beta(1-\beta)(1+2\beta) \right] \ , \qquad (3.16)$$

where $\alpha = a/R$. Also

$$W_f^{(3)} = - \frac{2\pi B_o^2}{\mu_o} c R^2 \alpha (1-\alpha) \frac{\beta}{1+2\beta} \exp\left[\frac{1-\beta}{\beta}\right] \int_{\frac{a}{c}\frac{1-\beta}{\beta}}^{\infty} \Phi_3 d\eta \ ,$$

where

$$\Phi_3 = (2+\alpha^3) \exp\left[-\frac{c\eta}{a}\right] - \frac{3\beta}{1+2\beta} \alpha (1-\alpha+\alpha^2) \exp\left[\frac{1-\beta}{\beta} - \frac{2c\eta}{a}\right] \ ,$$

so that

$$W_f^{(3)} = - \frac{2\pi B_o^2}{\mu_o} \frac{a^2 R\beta}{1+2\beta} \left[(2+\alpha^3) - \frac{3\beta}{1+2\beta} \frac{\alpha(1-\alpha+\alpha^2)}{2} \right] . \qquad (3.17)$$

By adding $W_f^{(2)}$ and $W_f^{(3)}$ we obtain the total flux:

$$W_f = \frac{\pi\beta_o^2 a^3}{3\mu_o} \left[\frac{(4+12\beta+9\beta^2-\beta^3)}{(1+2\beta)^2(1-\beta)} + \alpha^3 \right] . \qquad (3.18)$$

W_f thus contains two terms, one independent of R which is the total radiated energy W_r and a second term which varies as $1/R^3$, but is independent of β, and which can be interpreted as representing the field energy which is pushed out from one region in space to another. In other words, only part of the flux through a sphere of radius R represents radiation, the remaining part of the flux is energy which is removed from the field inside this sphere to be stored in the field outside it. This interpretation will be verified in the calculation of the field energy.

By putting $\alpha = 0$ in (3.18) we obtain the true radiated energy W_r:

$$W_r = \frac{4}{3} \frac{\pi B_o^2 a^3}{\mu_o} \frac{1+3\beta+\frac{9}{4}\beta^2 - \frac{1}{4}\beta^3}{(1+2\beta)^2(1-\beta)} . \qquad (3.19)$$

(b) Energy Extracted from the Field. The field components at $t = 0$ are

$$B_r = B_o \cos\theta$$

$$B_\theta = - B_o \sin\theta .$$

We can obtain their values at $t = \infty$ from Eq. (3.9) by putting $\eta = \infty$. They correspond to the fields surrounding a perfectly conducting sphere of radius $\underline{a}$ in a uniform field at infinity:

$$B_r = B_o \cos\theta(1 - a^3/r^3) \ ,$$
$$B_\theta = - B_o \sin\theta(1 + 2a^3/r^3) \ . \qquad (3.20)$$

The field energy density is given by $\frac{1}{2\mu_o}(B_r^2 + B_\theta^2)$, and although the total field energy for the above fields is infinite, the change in the total field energy can be calculated. Since we want to verify that the second term in Eq. (3.17) is actually a transfer of energy from one region to another, we find the change in field energy inside a sphere of radius R.

$$\Delta W = \frac{1}{2\mu_o}\int_o^{2\pi} d\varphi \int_o^{\pi} \sin\theta \, d\theta \int_o^R r^2 \, dr \{B^2_{final} - B^2_{initial}\} \ .$$

Since B_{final} is zero inside the conducting sphere, this splits into two integrals, one for $r > a$, and one for $r < a$. The second integral is $\frac{1}{2\mu_o} B_o^2$ multiplied by the volume of the sphere, $\frac{4}{3}\pi a^3$. Hence

$$\Delta W = \frac{\pi B_o^2}{\mu_o}\int_o^{\pi} \sin\theta \, d\theta \int_a^R \left[\frac{a^3}{R^3}(2\cos^2\theta + \sin^2\theta) + \frac{a^6}{R^6}(\cos^2\theta + \frac{1}{4}\sin^2\theta)\right] r^2 dr - \frac{2\pi a^3 B_o^2}{3\mu_o} \ ,$$

$$= - \frac{\pi a^3 B_o^2}{3\mu_o} - \frac{\pi a^3 B_o^2 \alpha^3}{\mu_o} \ , \qquad (3.21)$$

where $\alpha = a/R$ as before. Thus we see that the net change in field energy is negative and given by

$$\Delta W_t = - \frac{\pi a^3 B_o^2}{3\mu_o} \tag{3.22}$$

while the second term in (3.21) exactly balances the second term in the expression for the energy flux.

(c) Mechanical Work. The mechanical work done in expanding the sphere W_w can be calculated by conservation of energy:

$$W_w = W_r + \Delta W_t ,$$

$$= \frac{\pi a^3 B_o^2}{\mu_o} \frac{(1+\beta)^3}{(1+2\beta)^2(1-\beta)} \quad . \tag{3.23}$$

It should be noted that the same answer would result from a more local conservation of energy, namely

$$W_w = W_f + \Delta W \; .$$

Since electrodynamic problems usually do not involve moving boundaries on which the surface stresses may do work, it is instructive to calculate the mechanical work directly. This is most easily done by considering the surface stresses in the proper frame of the conductor. In this frame the Poynting vector is zero at the surface of the sphere, and the only force on the surface arises from the Maxwell stress tensor, there being no radiation reaction.

Consider an element of surface on which we attach a local

Cartesian system of coordinates with the (x', y', z') axes along the (r, θ, φ) directions respectively. In this frame the electric field is zero, and the magnetic field has only one component, B_y' which is equal to B_θ' outside the conductor and zero inside. The stress tensor will thus have different values inside and outside the conductor. The net force on the surface element is

$$d\vec{F}' = T'_{out} \cdot d\vec{S}' - T'_{in}\, dS$$

$$= \vec{e}_x (T'_{xx} - 0)\, dS = -\vec{e}_x \frac{B_y'^2}{2\mu_o}\, dS$$

and the pressure on it is

$$p' = \frac{B_y'^2}{2\mu_o} .$$

Now pressure is a four-scalar, while B_y' and E_z' transform as follows:

$$B_y' = \gamma(B_y + vE_z/c^2)$$

$$E_z' = \gamma(E_z + vB_y) ,$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2} .$$

But $E_z' = 0$ and therefore $E_z = -vB_y$ as we saw previously. Thus

$$B_y' = \gamma \left[B_y - \frac{v^2}{c^2} B_y \right] = B_y/\gamma ,$$

and

$$p = p' = \frac{1}{2\mu_o} \frac{B_y^2}{\gamma^2} = \frac{1}{2\mu_o} B_y^2 (1 - \beta^2) .$$

Thus, while the sphere is expanding, the pressure on its surface is

$$p = \frac{9B_o^2}{8\mu_o} \frac{(1 + \beta)^3}{(1 - \beta)(1 + 2\beta)^2} \sin^2\theta ,$$

where we have made use of the expression obtained for the field at the surface, Eq. (3.10).

The work done in expanding is thus

$$W_w = \int_o^V p\, dV = \int_o^{2\pi} d\varphi \int_o^{\pi} \sin\theta\, d\theta \int_o^a r^2\, dr \frac{9B_o^2}{8\mu_o} \sin^2\theta \frac{(1+\beta)^3}{(1-\beta)(1+2\beta)^2}$$

$$= \frac{\pi a^3 B_o^2}{\mu_o} \frac{(1+\beta)^3}{(1-\beta)(1+2\beta)^2} ,$$

as before. In Fig. 2 the work done is plotted as a function of β.

Some interesting points have appeared in this energy balance. One important point is that in spite of the fact that the presence of a sphere in the field decreases the field energy, work must still be done to put the sphere there. This is true even in the case of expansion at infinitely small speeds, and appears in the non-relativistic treatment given in Ref. 2. Another result which should be noticed is that the work done and the amount of energy radiated increase rapidly with the speed of expansion.

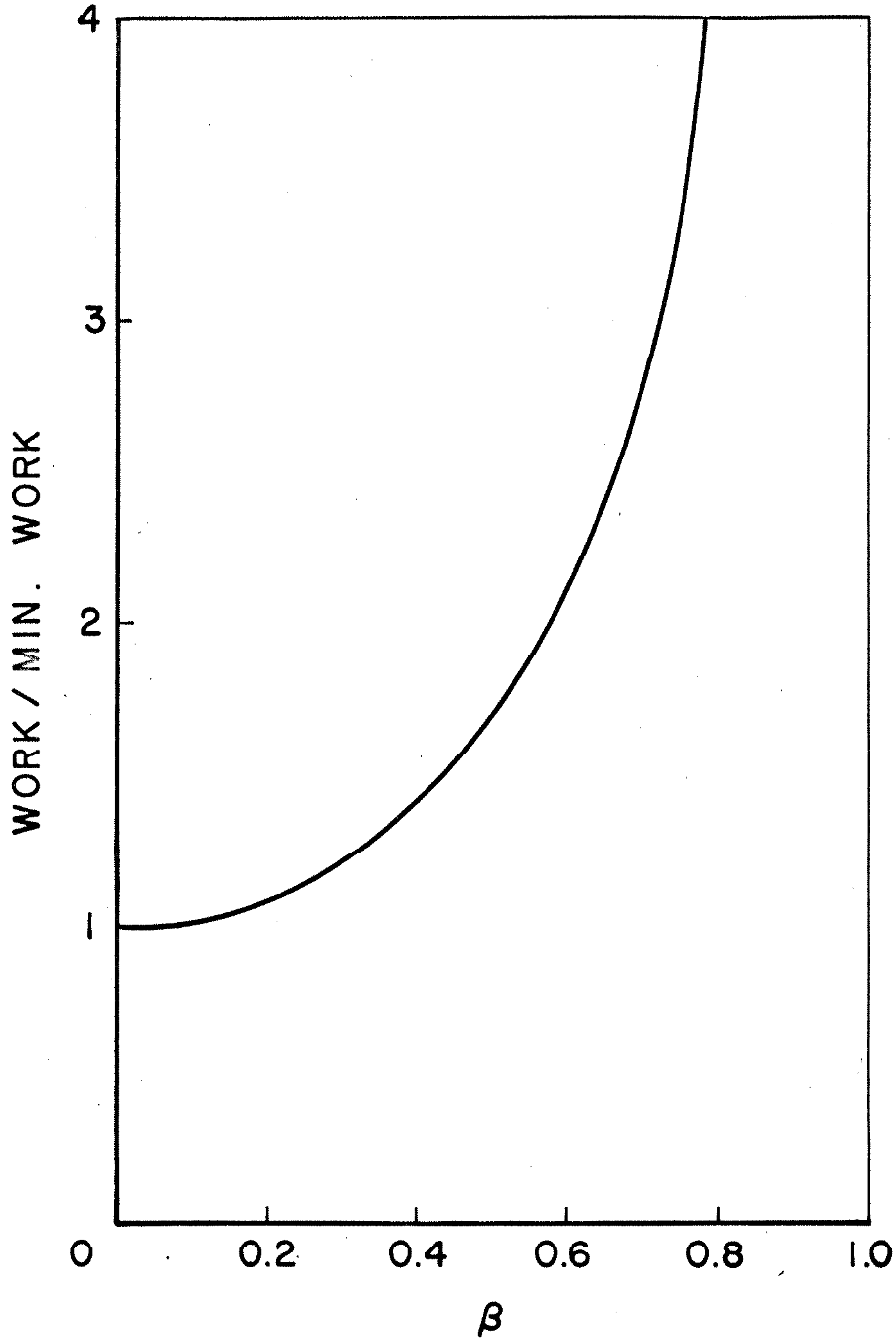

Figure 2. Work done by sphere in the expansion as a function of speed.

4. UNIFORMLY COLLAPSING SPHERE

4.1 Some Paradoxes

If one tries to apply the results of the previous section to a collapsing sphere, the conclusions are perplexing. The calculation of the field energy is certainly valid when applied to the problem of a sphere which starting from a radius $\underline{a}$ shrinks to zero radius. So that if in the previous case the energy decreased, in this case it must increase by the same amount. Furthermore, the Maxwell stresses are still acting to push the sphere inward, and, since in this case the walls yield, work is done on the sphere. The radiation condition still requires outgoing waves, to satisfy causality. Where then is all this energy coming from?

The resolution of this paradox illustrates another unusual situation in electrodynamics. We shall find that whereas waves are in fact travelling outwards, energy is travelling in. This is possible because the field contains an inexhaustible supply of energy, so that the waves in their outward motion constitute an outward rush to borrow energy, a debt which is repaid as the wave borrows energy from more and more remote regions. This cascading borrowing ends when the bill is collected at infinity, presumably as work done by the currents which maintain the uniform field.

Another question arises from this explanation. We have seen that in expansion energy is radiated, and anticipating the results that follow, we know that during collapse energy is drawn in from infinity. By expanding slowly, the radiated energy is minimized, although it cannot be made zero. Is it not possible then, that at some speed of collapse, (say by collapsing rapidly), the energy drawn in will be greater than that required to expand the sphere to its original radius? And by repeating this process, could we not remove all the energy from the field?

It will be seen in the sections that follow that the answer to both questions is in the negative. The amount of work done on the sphere decreases with the speed of collapse, and thus an oscillating sphere would radiate rather than extract energy from the magnetic field.

4.2 Fields

The problem considered in this section is that of a sphere of radius $\underline{a}$ which starting at $t = 0$ collapses at a uniform speed $v = \beta c$ until its radius is zero. In Fig. 3 the path described by the surface of the sphere is shown in the $r - t$ plane as the line MN. As in the previous case, the characteristics NQ and MP, together with the line MN, divide the plane into four regions. In region I the static field which was present initially is unaffected. The characteristic NQ carries the information that the sphere is collapsing, and the fields in region II reflect this fact. Similarly the characteristic MP carries the information that it has stopped contracting (in fact, that the sphere has disappeared). The fields present after this news has arrived are those in region III.

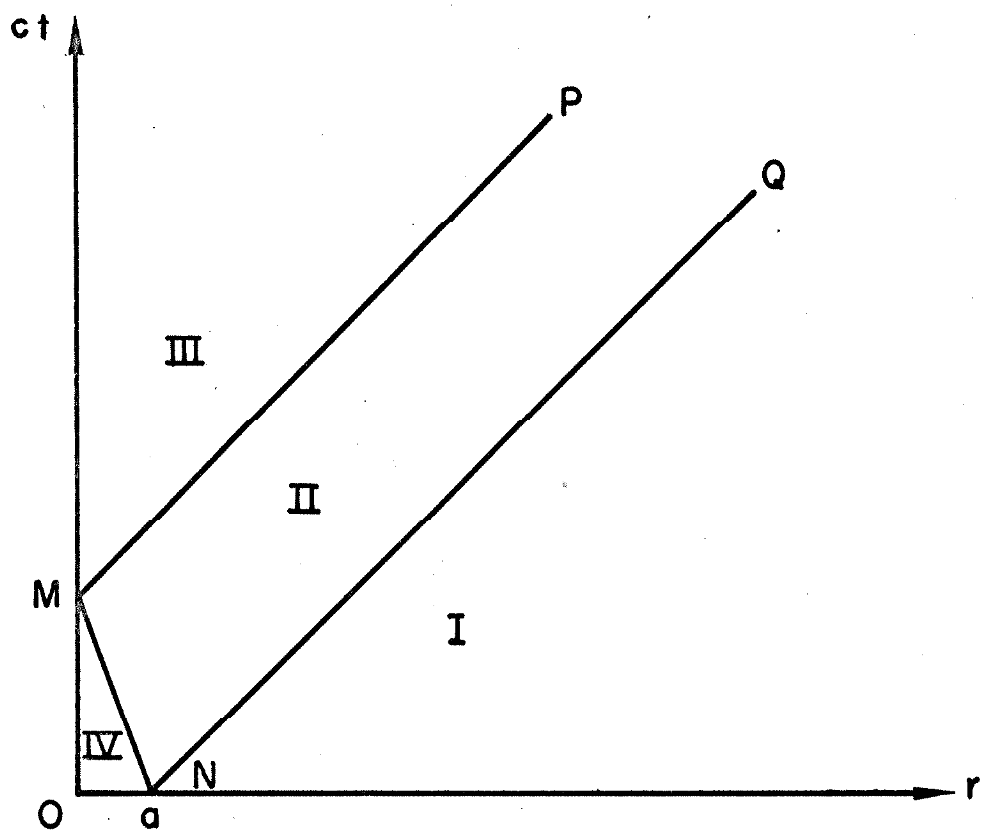

Figure 3. r - ct plane for the collapsing sphere.

In region I the vector potential is

$$A_{\varphi}^{(1)} = \frac{1}{2} B_o r \sin\theta(1 - a^3/r^3) \tag{4.1}$$

which gives rise to the fields given in Eq. (3.20). In region II, as discussed previously, A_{φ} will be of the form

$$A_{\varphi}^{(2)} = \frac{B_o}{2}\left[r - \frac{f(\eta) + \frac{r}{c} f'(\eta)}{r^2}\right] \sin\theta\ , \tag{4.2}$$

where η is now defined by $\eta = t - (r-a)/c$, which differs slightly from the previous definition.

The two expressions for A_{φ} must match along the boundary between regions I and II, which is not a physical surface. This is the line $ct = r - a$ or $\eta = 0$. Therefore

$$f(0) = a^3\ , \qquad f'(0) = 0\ . \tag{4.3}$$

These will be the initial conditions required to determine $f(\eta)$ from the differential equation.

This differential equation is obtained by applying the boundary condition $B_r = 0$ at the surface of the sphere, $t = (a-r)/v$. This requires that

$$r - \frac{f(\eta) + \frac{r}{c} f'(\eta)}{r^2} = 0 \quad \text{at} \quad r = a - \frac{\beta c\eta}{1+\beta}$$

or

$$f'(\eta) + \frac{1}{\frac{a}{c} - \frac{\beta\eta}{1+\beta}} f(\eta) = c^3 \left[\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right]^2 . \tag{4.4}$$

We can see that the two conditions (4.3) are consistent with this equation, for, if we put $\eta = 0$, and replace $f(0)$ by a^3 and $f'(0)$ by 0, Eq. (4.4) becomes an identity. This guarantees that if the initial condition $f(0) = a^3$ holds, the second condition will be satisfied automatically.

The solution of the equation is

$$f(\eta) = c^3 \left(\exp - \int \left[\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right]^{-1} d\eta \right) \int \left(\exp \int \left[\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right]^{-1} d\eta \right) \left(\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right)^2 d\eta,$$

$$= \frac{c^3(1+\beta)}{1-2\beta} \left[\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right]^3 + Nc^3 \left[\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right]^{(1+\beta)/\beta}$$

where N is a constant of integration. From Eq. (4.3),

$$N = - \frac{3\beta}{1-2\beta} \left(\frac{c}{a} \right)^{(1-2\beta)/\beta}$$

so that

$$f(\eta) = \frac{c^3}{1-2\beta} \left[(1+\beta) \left[\frac{a}{c} - \frac{\beta\eta}{1+\beta} \right]^3 - 3\beta \left[1 - \frac{\beta c\eta}{a(1+\beta)} \right]^{(1+\beta)/\beta} \right]. \tag{4.5}$$

In region III, A_φ must be of the form

$$A_\varphi^{(3)} = \frac{1}{2} B_o \left[\frac{g(\eta) + \frac{r}{c} g'(\eta)}{r^2} \right] \sin\theta , \tag{4.6}$$

where we again take $\eta = t - (r-a)/c$. Since A_φ must be continuous across the boundary between regions II and III, we must have

$A_\varphi^{(2)} = A_\varphi^{(3)}$ on the line $t = a/v + r/c$ or $\eta = a(1+\beta)/\beta c$. On this line $f(\eta) = 0$ and $f'(\eta) = 0$ so that

$$g\left(\frac{a(1+\beta)}{\beta c}\right) = g'\left(\frac{a(1+\beta)}{\beta c}\right) = 0 \ . \tag{4.7}$$

The further requirement that the fields remain finite at $r = 0$ makes $g(\eta) = 0$ and thus

$$A_\varphi^{(3)} = \frac{1}{2} B_o r \sin\theta \ , \tag{4.8}$$

which means that the fields are $\vec{B} = B_o \vec{e}_z$, $\vec{E} = 0$ throughout region III. This is a rather surprising result, for it might have been expected that, as in the problem of expansion, the fields would decay to this final value. We see instead that they reach this value immediately upon arrival at the signal carrying the information that the sphere has disappeared. This means that the information that the sphere was going to disappear was carried in the signals emanating from the surface throughout the period of collapse. Had the contraction not been carried to completion, however, we would again have obtained decaying fields in region III, for in this case the information that the sphere was collapsing would suddenly have had to be corrected to say that the collapse was incomplete.

The only time dependent fields in this problem are those in region II. From the expression (4.5) for $f(\eta)$ we can derive the field components for this region. These are rather lengthy expressions, so we shall write them in terms of $f(\eta)$ and its derivatives, and list these below:

$$B_r = B_o \cos\theta \left[1 - \frac{f(\eta)}{r^3} - \frac{f'(\eta)}{r^2 c} \right]$$

$$B_\theta = -\frac{1}{2} B_o \sin\theta \left[2 + \frac{f(\eta)}{r^3} + \frac{f'(\eta)}{cr^2} + \frac{f''(\eta)}{c^2 r} \right] , \qquad (4.9)$$

$$E_\varphi = \frac{1}{2} B_o \sin\theta \left[\frac{f'(\eta)}{r^2} + \frac{f''(\eta)}{rc} \right]$$

where

$$f(\eta) = \frac{c^3}{1-2\beta} \left[(1+\beta)\tau^3 - 3\beta \left(\frac{c}{a}\right)^{(1-2\beta)/\beta} \tau^{(1+\beta)/\beta} \right]$$

$$f'(\eta) = \frac{3\beta c^3}{1-2\beta} \left[-\tau^2 + \left(\frac{c}{a}\right)^{(1-2\beta)/\beta} \tau^{1/\beta} \right] \qquad (4.10)$$

$$f''(\eta) = \frac{3\beta c^3}{(1-2\beta)(1+\beta)} \left[2\beta\tau - \left(\frac{c}{a}\right)^{(1-2\beta)/\beta} \tau^{(1-\beta)/\beta} \right]$$

and

$$\tau = \frac{a}{c} - \frac{\beta\eta}{1+\beta} .$$

It should be noted that the expressions given for the field components in Eq. (4.9) are valid for all problems in which A_φ is of the form (4.2).

4.3 Energy Balance

With these expressions for the field quantities we are now in a position to calculate the radiated energy and work done. Since we did these calculations in detail in the previous problem, there is nothing to be gained in going over the procedure a second time. We shall therefore limit ourselves to giving the results.

As discussed previously, the radiated energy is negative:

$$W_r = -\frac{4\pi a^3 B_o^2}{3\mu_o} \frac{2+3\beta - \frac{9}{2}\beta^2 + \frac{1}{2}\beta^3}{(1+\beta)^2(2-\beta)} \quad . \tag{4.11}$$

The change in field energy is the same as in the case of expansion, only opposite in sign:

$$\Delta W_t = \frac{\pi a^3 B_o^2}{3\mu_o} \quad . \tag{4.12}$$

The total mechanical work is the sum of these

$$\begin{aligned} W_w &= W_r + \Delta W_2 \\ &= -\frac{\pi a^3 B_o^2}{\mu_o} \frac{(1-\beta)(2+5\beta-\beta^2)}{(1+\beta)^2(2-\beta)} \quad . \end{aligned} \tag{4.13}$$

This result is plotted in Fig. 4, from which it can be seen that the work done on the sphere is a maximum when $\beta = 0$ and decreases to zero as β increases towards 1. The energy extracted from the field at $\beta = 0$ is equal to the work that is done by an expanding sphere at that speed. Thus we can conclude that a cyclic process will result in a net loss of energy through radiation unless it is performed at an infinitely slow speed. The amount radiated will increase rapidly with the speed of the process. In the next section the fields due to a cyclic expansion and contraction of a sphere are derived.

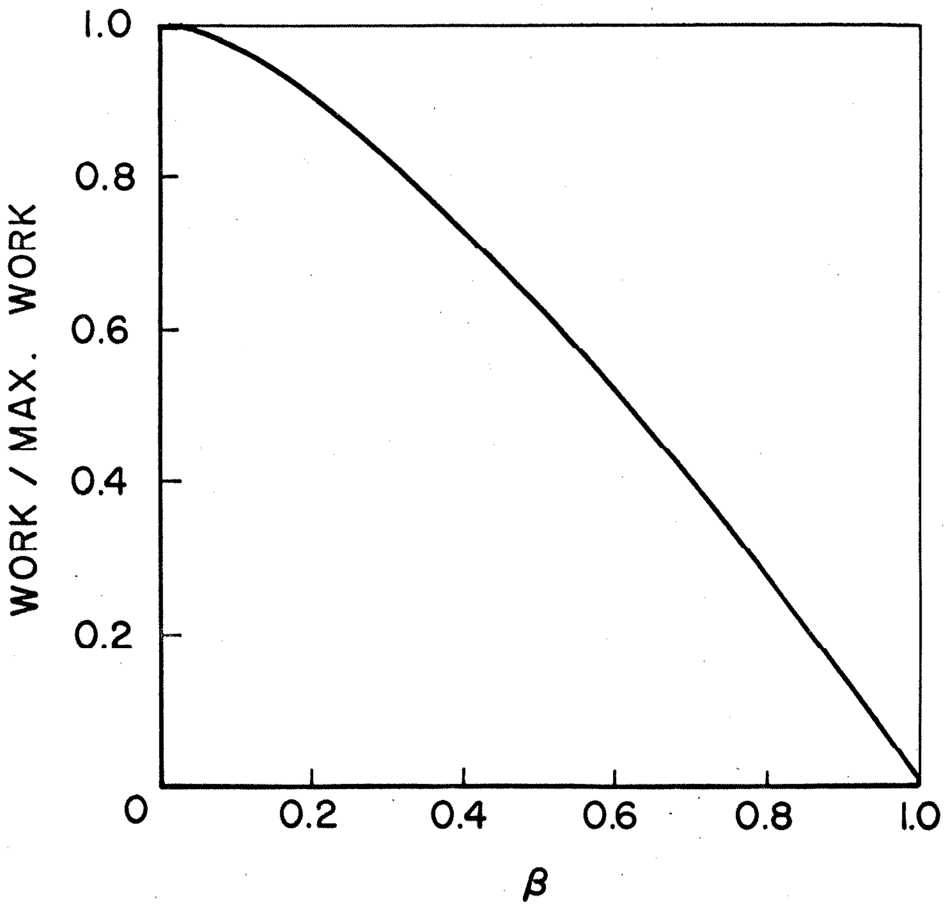

Figure 4. Work done on the collapsing sphere as a function of speed.

5. OSCILLATING SPHERE

5.1 Fields

We shall now consider the problem of a sphere which oscillates between two radii a and b by successive expansions and contractions done at a constant speed $v = \beta c$. We shall assume that the process is repeated periodically and that all transients have died out; that is, we seek a periodic solution to the problem.

The motion of the surface is shown in Fig. 5. The sphere expands from radius a to radius b along the line PQ, and returns to its original radius along QR, at which time a new cycle begins. The condition of periodicity requires that the conditions that exist along the characteristic PL must be duplicated exactly along RN. Thus the r-t plane is divided into three basic regions. Region I is adjacent to the path that the surface follows on expansion, region II to the path it follows on contraction, and region III represents the interior of the sphere. Region II is followed by a region which is identical to region I, just as region I follows one that is identical to II.

As before, we can put

$$A_\varphi^{(1)} = \frac{1}{2} B_o \left[r - \frac{f(\eta) + \frac{r}{c} f'(\eta)}{r^2} \right] \sin\theta , \qquad (5.1)$$

and

$$A_\varphi^{(2)} = \frac{1}{2} B_o \left[r - \frac{g(\eta) + \frac{r}{c} g'(\eta)}{r^2} \right] \sin\theta , \qquad (5.2)$$

where $\eta = t - (r-a)/c$. The differential equations for $f(\eta)$ and $g(\eta)$

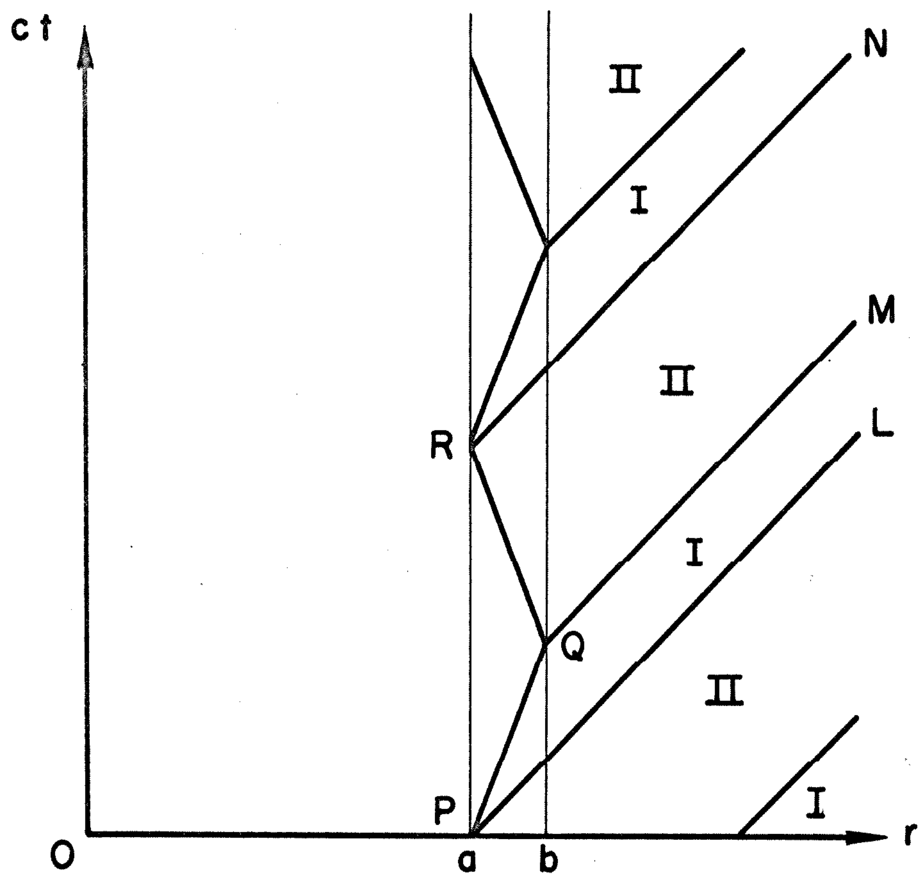

Figure 5. r - ct plane for the oscillating sphere.

are obtained by applying the conditions $B_r = 0$ on PQ and QR. The functions are then determined uniquely by putting $A_\varphi^{(1)} = A_\varphi^{(2)}$ along QM and requiring that $A_\varphi^{(1)}$ along PL by the same as $A_\varphi^{(2)}$ along RN.

This leads to the following system of equations:

$$f'(\eta) + \frac{c}{r} f(\eta) = cr^2 \quad \text{along} \quad r = a + \frac{\beta c \eta}{1-\beta} , \tag{5.3}$$

$$g'(\eta) + \frac{c}{r} g(\eta) = cr^2 \quad \text{along} \quad r = a + \frac{1}{1+\beta}[2(b-a) - \beta c \eta] ; \tag{5.4}$$

subject to the conditions

$$f\left(\frac{(b-a)(1-\beta)}{\beta c}\right) = g\left(\frac{(b-a)(1-\beta)}{\beta c}\right) , \tag{5.5}$$

$$f(0) = g\left(\frac{2(b-a)}{\beta c}\right) . \tag{5.6}$$

The solutions to Eqs. (5.3) and (5.4) are

$$f(\eta) = \left(K_1 \left[a + \frac{\beta c \eta}{1-\beta}\right]^{-(1-\beta)/\beta} + \frac{1-\beta}{1+2\beta}\left[a + \frac{\beta c \eta}{1-\beta}\right]^3\right), \tag{5.7}$$

and

$$g(\eta) = \left(K_2 \left[a' - \frac{\beta c \eta}{1-\beta}\right]^{(1+\beta)/\beta} + \frac{1-\beta}{1-2\beta}\left[a' - \frac{\beta c \eta}{1+\beta}\right]^3\right), \tag{5.8}$$

where

$$a' = a + 2(b-a)/(1+\beta) .$$

The constants of integration K_1 and K_2 must be evaluated by applying conditions (5.5) and (5.6). These give rise to the equations

$$K_1 b^{-(1-\beta)/\beta} + \frac{1-\beta}{1+2\beta} b^3 = K_2 b^{(1+\beta)/\beta} + \frac{1+\beta}{1-2\beta} b^3$$

and

$$K_1 a^{-(1-\beta)/\beta} + \frac{1-\beta}{1+2\beta} a^3 = K_2 a^{(1+\beta)/\beta} + \frac{1+\beta}{1-2\beta} a^3 ,$$

from which we obtain

$$K_1 = \frac{6\beta}{1-4\beta^2} \frac{a^{-(1-2\beta)/\beta} - b^{-(1-2\beta)/\beta}}{a^{-2/\beta} - b^{-2/\beta}} , \tag{5.9}$$

$$K_2 = \frac{6\beta}{1-4\beta^2} \frac{b^{(1+2\beta)/\beta} - a^{(1+2\beta)/\beta}}{b^{2/\beta} - a^{2/\beta}} . \tag{5.10}$$

The field components can then be obtained from Eq. (4.9)

5.2 Radiation

It is of interest to find the amount of energy radiated per cycle by the oscillating sphere. From Eq. (4.9) the radial component of the Poynting vector is, in terms of $f(\eta)$,

$$S_r = \frac{B_o^2}{4\mu_o} \sin^2\theta \left[\frac{2f''}{cr} + \frac{1}{r^2}\left(\frac{f''^2}{c^3} + 2f'\right) + \frac{2f'f''}{c^2 r^3} + \frac{1}{cr^4}\left(f'^2 + ff''\right) + \frac{ff'}{r^5} \right] . \tag{5.11}$$

In computing the radiated energy the terms which go to zero faster than $1/r^2$ drop out. In addition since the combined functions f and g are periodic, terms such as f' or f'' integrate out to zero when the integral extends over a full period. Thus we obtain for the energy radiated in one period, after making use of Eq. (3.15)

$$W_r = \frac{2\pi B_o^2}{2\mu_o c^3}\left[\int_o^{(b-a)(1-\beta)/\beta c} f''^2(\eta)\, d\eta + \int_{(b-a)(1-\beta)/\beta c}^{2(b-a)/\beta c} g''^2(\eta)\, d\eta\right] . \tag{5.12}$$

Making the substitution $x = a + \beta c\eta/(1-\beta)$ in the first integral,

$$I_1 = \int_o^{(b-a)(1-\beta)/\beta c} f''^2(\eta)\, d\eta \quad ,$$

$$= \frac{\beta^3 c^3}{1-\beta}\int_a^b\left[\frac{36}{(1+2\beta)^2}\, x^2 + \frac{12K_1}{\beta^2(1+2\beta)}\, x^{-1/\beta} + \frac{K_1^2}{\beta^4}\, x^{-2(1+\beta)/\beta}\right] dx \, ,$$

$$= \frac{\beta^3 c^3}{1-\beta}\left[\frac{12}{(1+2\beta)^2}\,(b^3-a^3) - \frac{12K_1}{\beta(1+2\beta)(1-\beta)}\left[b^{-(1-\beta)/\beta} - a^{-(1-\beta)/\beta}\right]\right.$$

$$\left. - \frac{-K_1^2}{\beta^3(2+\beta)}\left[b^{-(2+\beta)/\beta} - a^{-(2+\beta)/\beta}\right]\right] . \tag{5.13}$$

Similarly the substitution on $x = a' - \beta c\eta/(1+\beta)$ transforms the second integral into

$$I_2 = \int_{(b-a)(1-\beta)/\beta c}^{2(b-a)/\beta c} g''^2(\eta)\, d\eta \quad ,$$

$$= \frac{\beta^3 c^3}{1+\beta}\int_a^b\left[\frac{36}{(1-2\beta)^2}\, x^2 + \frac{12K_2}{\beta^2(1-2\beta)}\, x^{1/\beta} + \frac{K_2^2}{\beta^4}\, x^{2(1-\beta)/\beta}\right] dx \, ,$$

$$= \frac{\beta^3 c^3}{1+\beta}\left[\frac{12}{(1-2\beta)^2}\left[b^3-a^3\right] + \frac{12K_2}{\beta(1-2\beta)(1+\beta)}\left[b^{(1+\beta)/\beta} - a^{(1+\beta)/\beta}\right]\right.$$

$$\left. + \frac{K_2^2}{\beta^3(2-\beta)}\left[b^{(2-\beta)/\beta} - a^{(2-\beta)/\beta}\right]\right] . \tag{5.14}$$

These expressions can be written in a simpler form if we put

$$b = R(1 + \epsilon) \quad , \qquad a = R(1 - \epsilon) \tag{5.15}$$

and define

$$D(x) = (1 + \epsilon)^x - (1 - \epsilon)^x \quad . \tag{5.16}$$

Then

$$\begin{aligned} I_1 &= \frac{12\beta^3 c^3 R^3}{(1-\beta)(1+2\beta)^2} \; F(\beta, \epsilon) \\ I_2 &= \frac{12\beta^3 c^3 R^3}{(1+\beta)(1-2\beta)^2} \; F(-\beta, \epsilon) \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} F(\beta, \epsilon) = D(3) &- \frac{6}{(1-\beta)(1-2\beta)} \; \frac{D(2-1/6)D(1-1/\beta)}{D(-2/\beta)} \\ &- \frac{3}{\beta(1-2\beta)^2(2+\beta)} \; \frac{D^2(2-1/\beta)D(1-2/\beta)}{D^2(-2/\beta)} \quad . \end{aligned} \tag{5.18}$$

The energy radiated per unit time is

$$P = \frac{2\pi B_o^2 R^2 c}{\mu_o} \; \frac{\beta^4}{\epsilon} \left[\frac{1}{(1-\beta)(1+2\beta)^2} F(\beta, \epsilon) + \frac{1}{(1+\beta)(1-2\beta)^2} F(-\beta, \epsilon) \right] . \tag{5.19}$$

In Fig. 6 this is plotted as a function of β with ϵ as a parameter. The power is divided by

$$P_o = 2\pi B_o^2 R^2 c / \mu_o \tag{5.20}$$

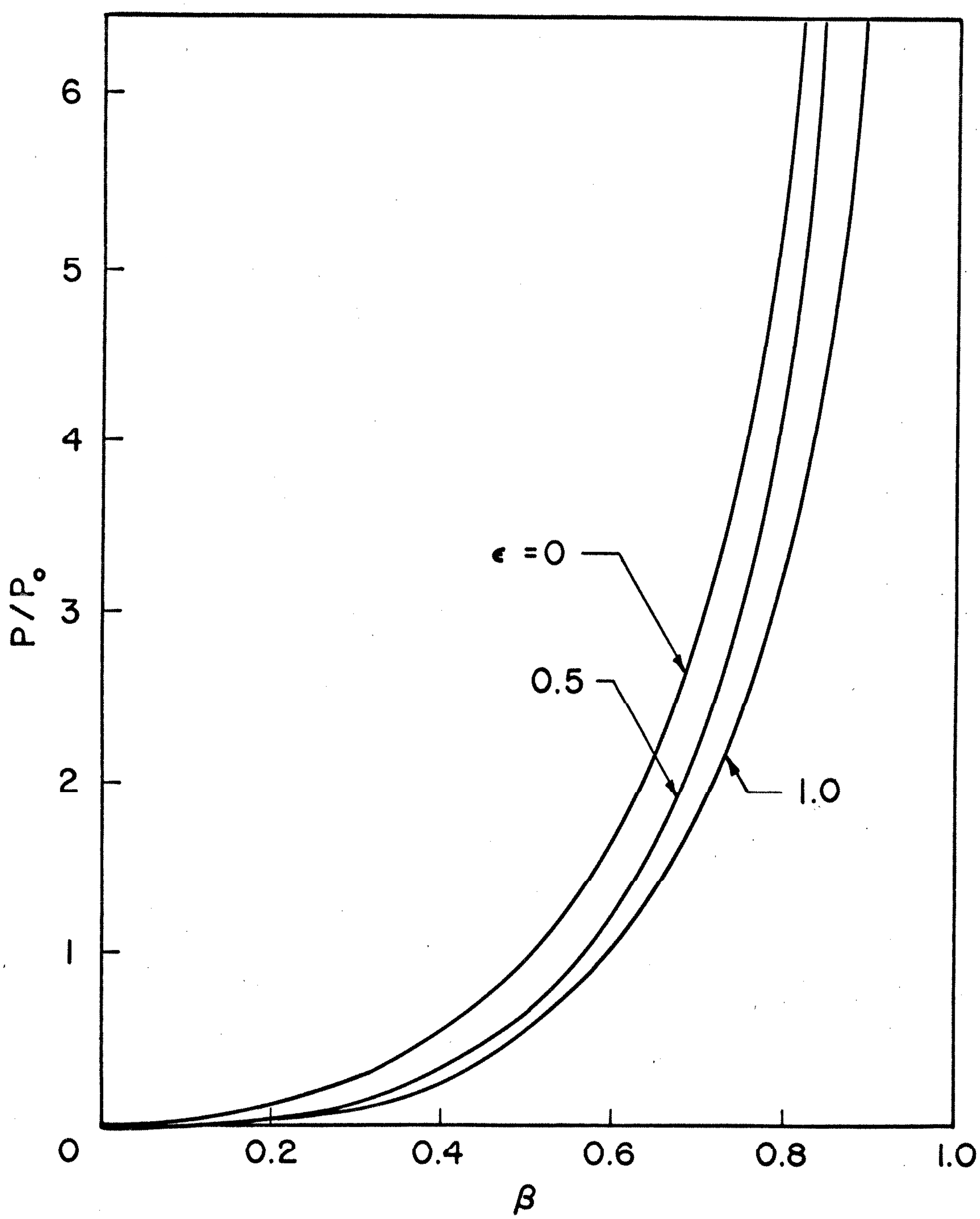

Figure 6. Radiated power as a function of speed for various amplitudes.

to make it dimensionless.

An interesting special case is obtained by assuming that the oscillations are small, about a mean radius R. This limit must be taken carefully, since the result will depend on whether other variables go to zero together with ϵ. We shall take the limit letting both ϵ and β go to zero together in such a way that the frequency remains constant.

The frequency of the oscillations is

$$\omega = \frac{\pi\beta c}{2\epsilon R} \tag{5.21}$$

and

$$\beta/\epsilon = \frac{2\omega R}{\pi c} .$$

We thus have to expand $D(x)$ for ϵ and β approaching zero while β/ϵ stays constant.

Since

$$(1+\epsilon)^{\lambda/\epsilon} = e^{\lambda} + O(\epsilon) ,$$

we obtain

$$D(n+\lambda/\epsilon) = 2 \sinh \lambda .$$

Making use of this approximation and the definition of ω we obtain

$$P = \frac{48 B_o^2 \omega^3 R^5}{\mu_o \pi^2 c^2} \epsilon^2 \tanh \frac{\pi c}{2\omega R} ,$$

$$= 3P_o \left[\frac{2\omega R}{\pi c}\right]^3 \epsilon^2 \tanh \frac{\pi c}{2\omega R} . \tag{5.22}$$

The value of $P/P_o\epsilon^2$ is plotted in Fig. 7 as a function of β/ϵ. It is also plotted for oscillations of finite amplitude. The figure shows that the effect of larger amplitudes is to increase the radiated power at any given frequency of oscillation, the effect being more pronounced at higher frequencies.

Other limiting values of the radiated power may be of interest. If we let the amplitude ϵ approach zero, keeping β constant, we obtain

$$P = \frac{3P_o\beta^2}{1-\beta^2} \quad . \tag{5.23}$$

This represents the curve labelled $\epsilon = 0$ in Fig. 6.

If, on the other hand, we let β approach zero with ϵ constant, the limiting value of P is

$$P = 3P_o\beta^3(1 + 3\epsilon^2)/\epsilon \quad . \tag{5.24}$$

Equation (5.24) in the limit of small ϵ agrees with Eq. (5.22) only if ω is assumed to be small, because this assumption was made in obtaining Eq. (5.24). In the next section the solution to the linearized problem will be obtained. By comparing the results that we have just obtained with those that follow, one must bear in mind that the linearized solution is carried out for finite values of the frequency. Thus Eq. (5.22) is the relevant one for this comparison.

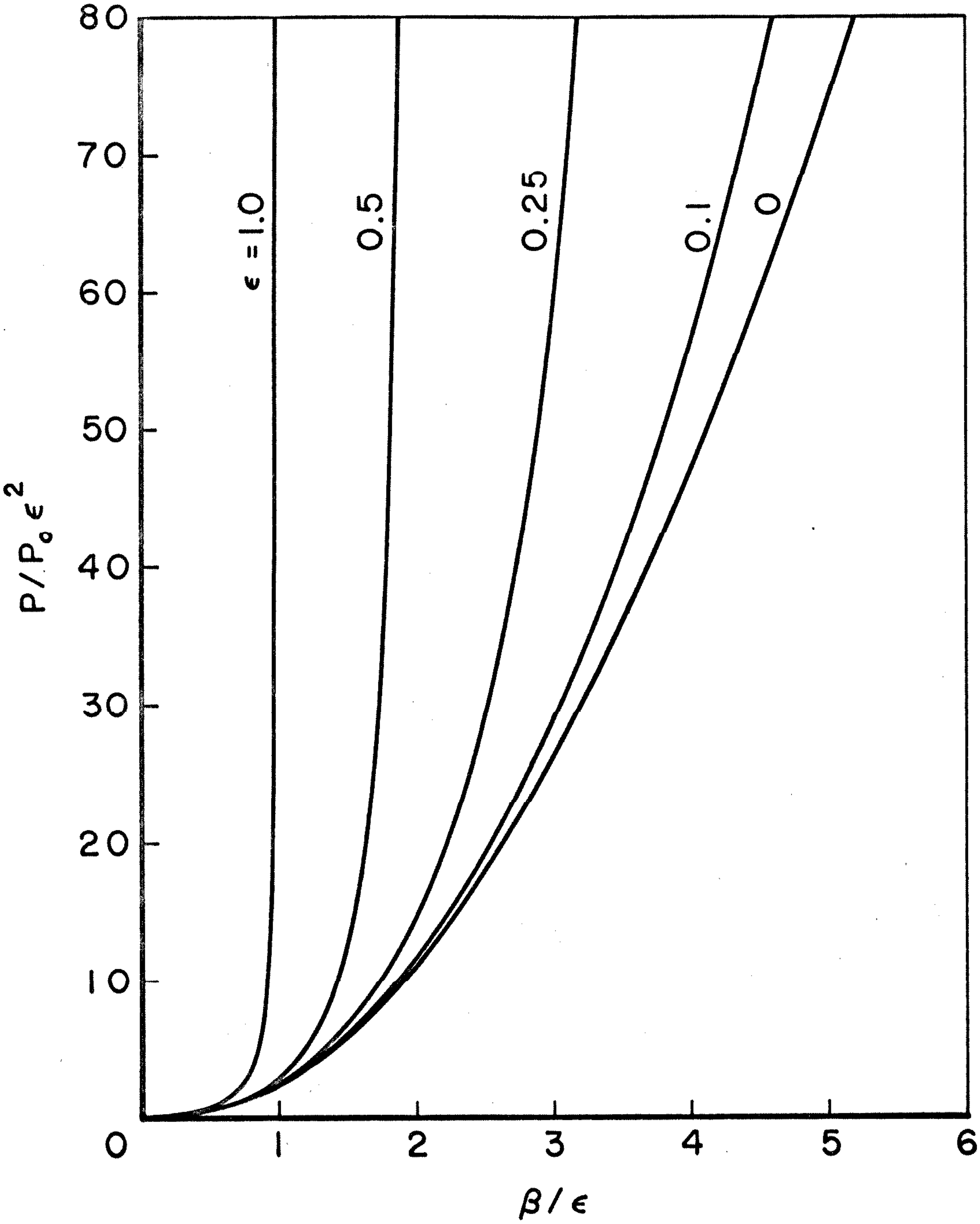

Figure 7. Radiated power/(Amplitude)^2 as a function of frequency.

5.3 Small Amplitude Oscillations

If the boundary conditions are linearized, it is possible to obtain a solution for small amplitude sinusoidal oscillations of a sphere of radius R. By Fourier analysis this solution can be used to obtain the fields due to arbitrary small amplitude motions of the surface. In the next section we will also obtain a first order correction to the fields due to a sinusoidal oscillation. The correction, however, will apply only to the case worked out, and can only be used as an indication of the error involved in using small amplitude analysis.

The vector potential can be split into a steady and a time dependent part:

$$A_\varphi = A_\varphi^{(o)} + \frac{1}{2} B_o \sin\theta \left[\frac{f(\eta) + \frac{r}{c} f'(\eta)}{r^2} \right] . \qquad (5.25)$$

where

$$\eta = t - (r - R)/c$$

and

$$A_\varphi^{(o)} = \frac{1}{2} B_o r \sin\theta \,(1 - R^3/r^3) \qquad (5.26)$$

is the static field outside a sphere of radius R.

The boundary condition $B_r = 0$ applied to Eq. (5.25) gives

$$f(\eta) + \frac{r}{c} f'(\eta) = R^3 - r^3 . \qquad (5.27)$$

Let the surface of the sphere move according to the equation

$$r = R(1 - \epsilon \cos \omega t)$$

then to first order in ϵ, Eq. (5.27) becomes

$$f(t) + \frac{R}{c} f'(t) = 3\epsilon R^3 \cos \omega t \tag{5.28}$$

so that

$$f(t) = \frac{3\epsilon R^3}{1+k^2R^2} (\cos \omega t + kR \sin \omega t) \tag{5.29}$$

where

$$k = \omega/c \ .$$

To calculate the radiated energy, we make use of Eq. (5.11). Since f is a periodic function, only one term contributes to the integral, and the energy radiated per cycle is

$$W_r = \frac{2\pi B_o^2}{3\mu_o c^3} \int_o^{2\pi/\omega} f''^2(t) \, dt = \frac{6\pi^2 B_o^2 \epsilon^2 k^3 R^6}{\mu_o(1+k^2R^2)} \ ,$$

and the energy radiated per unit time is

$$P = \frac{3\pi B_o^2 c k^4 R^6 \epsilon^2}{\mu_o(1+k^2R^2)} \ . \tag{5.30}$$

To compare this with the answer obtained in the previous section, we must find the power radiated by a sphere whose surface moves in a triangular, instead of a sinusoidal wave. The motion of the surface is described by the equation

$$r = R - \frac{8\epsilon R}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)\omega t]}{(2n+1)^2} \ .$$

From Eq. (5.29)

$$f(t) = \frac{24\epsilon R^3}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos[(2n+1)\omega t] + (2n+1)kR\sin[(2n+1)\omega t]}{[1+(2n+1)^2k^2R^2](2n+1)^2}$$

and from Eq. (5.30)

$$P = \frac{192 B_o^2 k^4 c\epsilon^2 R^6}{\pi^3 \mu_o} \sum_{n=0}^{\infty} \frac{1}{1+(2n+1)^2k^2R^2} .$$

This expression can be simplified by making use of the expansion

$$\tan\frac{\pi x}{2} = \frac{4x}{\pi} \sum_{n=0}^{\infty} \frac{1}{x^2+(2n+1)^2} ,$$

which changes P into

$$P = \frac{48 B_o^2 \omega^3 R^5 \epsilon^2}{\mu_o \pi^2 c^2} \tanh\frac{\pi c}{2\omega R}$$

as in Eq. (5.22).

5.4 Effect of Finite Amplitudes

To find the modifications that occur due to the finite size of the oscillations, we expand Eq. (5.27) to a second order in ϵ. To do this we must find r in terms of η to second order in ϵ.

Since

$$\eta = t + \frac{\epsilon R}{c}\cos\omega t ,$$

$$\cos\omega t = \cos(\omega\eta - \epsilon kR\cos\omega t) ,$$

$$= \cos(\epsilon kR\cos\omega t)\cos\omega\eta + \sin(\epsilon kR\cos\omega t)\sin\omega\eta .$$

To first order in ϵ this is

$$\cos \omega t = \cos \omega\eta + \epsilon kR \cos \omega\eta \sin \omega\eta + O(\epsilon^2) ,$$

so that

$$r = R - \epsilon R \cos \omega\eta - \epsilon^2 kR^2 \cos \omega\eta \sin \omega\eta + O(\epsilon^3) .$$

Equation (5. 25) becomes

$$f(\eta) + \frac{R}{c}(1 - \epsilon \cos \omega\eta)f'(\eta) = R^3[1 - (1 - \cos \omega\eta - \epsilon^2 kR \cos \omega\eta \sin \omega\eta)^3]$$

$$= 3\epsilon R^3 \cos \omega\eta [1 + \epsilon kR \sin \omega\eta - \epsilon \cos \omega\eta] .$$

This can be solved to the required order in ϵ by putting

$$f(\eta) = f_o(\eta) + \epsilon f_1(\eta) .$$

Then f_o obeys Eq. (5. 28) and is thus given by Eq. (5. 29) while f_1 satisfies

$$f_1(\eta) + \frac{R}{c} f_1'(\eta) = \frac{R}{c} \cos \omega\eta f_o'(\eta) + 3\epsilon R^3 \cos \omega\eta(kR \sin \omega\eta - \cos \omega\eta)$$

$$= 3\epsilon R^3 \cos \omega\eta \frac{k^3R^3 \sin \omega\eta - \cos \omega\eta}{1 + k^2R^2} ,$$

from which we obtain that

$$f_1(\eta) = -\frac{3}{2}\frac{\epsilon R^3}{1+k^2R^2}\left[1 + \frac{(1+2k^4R^4)\cos 2\omega\eta + kR(2 - k^2R^2)\sin 2\omega\eta}{1 + 4k^2R^2}\right] .$$

Since the sine and cosine terms are orthogonal when integrated over one period, there will be no cross terms between f_o and f_1 in the integral of f''^2. Thus we can compute the additional power radiated:

$$P_1 = \frac{12\pi B_o^2 k^4 R^6 c\epsilon^4 (1 + 4k^2R^2 + k^6R^6 + 4k^8R^8)}{\mu_o (1 + k^2R^2)^2 (1 + 4k^2R^2)^2} ,$$

so that the total power radiated is

$$P = \frac{3\pi B_o^2 k^4 R^6 c\epsilon^2}{\mu_o (1 + k^2R^2)} \left[1 + \frac{4\epsilon^2 (1 + 4k^2R^2 + k^6R^6 + 4k^8R^8)}{(1 + k^2R^2)(1 + 4k^2R^2)^2} \right] . \quad (5.31)$$

From which we see that the correction term is small if $\epsilon kR \ll 1$. This requirement implies that the speed of the surface must be small compared to c. This is in agreement with the results summarized in Fig. 7.

6. CONCLUSIONS

The three examples of spherical motion considered in the preceding pages bring out some interesting points regarding the energy present in a uniform field and the energy balance in processes involving the deformation of a body placed in a magnetic field. The results found are consequences of the fact that the applied field considered here does not die off at infinity, so that there is an infinite amount of energy stored in the field. This energy is available to do work and the field can thus become an energy reservoir from which energy can be extracted or borrowed temporarily.

In the example of an expanding sphere, the sphere does work on the field, and the field energy decreases. The sum of these energies is radiated and eventually absorbed by the sources which maintain the field. The work done on the field, and thus the contribution to the radiated energy from it, increases with the speed of expansion.

The contracting sphere gives rise to more curious phenomena. In this case the sphere extracts work from the field, the amount of work decreasing with the speed of contraction. The field energy also increases and thus energy is extracted from the sources at infinity. This extraction process cannot be instantaneous, however, because of the finite speed of propagation of the electromagnetic disturbance. The energy balance is maintained by an outward wave which borrows energy from the infinite energy present in the field and eventually recovers it from the sources. Thus, in this instance, the direction of energy travel is opposite to that of propagation of the disturbance. Another interesting result is obtained when the contraction is carried out to its foreseeable end, that is, until the sphere disappears. The fields then settle to their final values at the retarded time that corresponds to this end condition. It is only when an unpredictable change occurs in the motion of the sphere that the fields decay exponentially.

The oscillating sphere radiates energy upon expansion and recovers some of it on contraction, but the net effect is to radiate some energy in each cycle. The results obtained were useful in evaluating the accuracy of the linearized analysis, since the exact answer was found to the non-linear problem. Although some error was present, the correct behavior was obtained for the radiated power as a function of frequency. This would indicate that the linearized approximation may be used with confidence even for large amplitude oscillations in problems of this sort. The error made would be outweighed by the advantages of linear analysis, such as superposition of motions of the boundary. It should be kept in mind, however, that the motions considered in this study are highly symmetrical and this may have some bearing on the accuracy of the linearized approximation.

APPENDIX A

A.1 Solution of the Wave Equation for the Scalar Potentials

The scalar functions introduced in Eq. (1.12) satisfy the partial differential equation

$$\frac{1}{r}\frac{\partial^2(r\varphi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\varphi)\right] - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 . \qquad (A.1)$$

The eigenfunctions of the operator

$$\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\right]$$

are the associated Legendre polynomials of order 1, $P_n^1(\cos\theta)$, since

$$-\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\, P_n^1(\cos\theta)\right]\right) = n(n+1)P_n^1(\cos\theta) . \qquad (A.2)$$

Therefore if we put

$$\varphi = \sum \psi_n(r,t)\, P_n^1(\cos\theta), \qquad (A.3)$$

φ will be a solution of the above equation provided ψ_n satisfies

$$\frac{\partial^2\psi_n}{\partial r^2} + \frac{2}{r}\frac{\partial\psi_n}{\partial r} - \frac{n(n+1)}{r^2}\psi_n - \frac{1}{c^2}\frac{\partial^2\psi_n}{\partial t^2} = 0 . \qquad (A.4)$$

It will now be shown how the ψ_n can be generated. The result is similar to the recursion formula for spherical Bessel functions.

We first show that if ψ_n is a solution of (A.4), then

$$\psi_{n+1} = \frac{\partial \psi_n}{\partial r} - \frac{n}{r}\psi_n \tag{A.5}$$

is also a solution of that equation, with n replaced by $n+1$. To do this we operate on Eq. (A.4) with $\partial/\partial r - n/4$. The result is

$$\frac{\partial^3 \psi_n}{\partial r^3} - \frac{n-2}{r}\frac{\partial^2 \psi_n}{\partial r^2} - \frac{(n+1)(n+2)}{r^2}\frac{\partial \psi_n}{\partial r} + \frac{n(n+1)(n+2)}{r^2}\psi_n - \frac{1}{c^2}\frac{\partial \psi_{n+1}}{\partial t^2} = 0 \,. \tag{A.6}$$

If we also operate on Eq. (A.5) with $\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ and combine with Eq. (A.6) we obtain

$$\frac{\partial^2 \psi_{n+1}}{\partial r^2} + \frac{2}{r}\frac{\partial \psi_{n+1}}{\partial r} - \frac{(n+1)(n+2)}{r^2}\psi_{n+1} - \frac{1}{c^2}\frac{\partial^2 \psi_{n+1}}{\partial t^2} = 0 \tag{A.7}$$

which is the same as Eq. (A.4) with n replaced by $n+1$ throughout.

We now look for the solution which starts the chain, ψ_o. This function must satisfy the equation

$$\frac{\partial^2 \psi_o}{\partial r^2} + \frac{2}{r}\frac{\partial \psi_o}{\partial r} - \frac{1}{c^2}\frac{\partial^2 \psi_o}{\partial t^2} = 0. \tag{A.8}$$

Making the substitution

$$\chi = r\psi_o \,,$$

we find that χ obeys the one-dimensional wave equation,

$$\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2} = 0 \qquad (A.9)$$

which has the solutions

$$\chi = f_1(t - r/c) \; , \quad f_2(t + r/c)$$

the first corresponding to the retarded potential, the second to the advanced. If we limit ourselves to retarded solution,

$$\psi_o = \frac{f(t - r/c)}{r} \quad . \qquad (A.10)$$

An expression for ψ_n can be obtained by rewriting (A. 5) as follows:

$$\psi_{n+1} = r^n \frac{\partial}{\partial r}\left[\frac{1}{r^n}\psi_n\right] \qquad (A.11)$$

which after n applications yields

$$\psi_n = r^n\left[\frac{1}{r}\frac{\partial}{\partial r}\right]^n \frac{f(t - r/c)}{r} \quad . \qquad (A.12)$$

This is the desired result.

APPENDIX B

B.1 The Problem of Arbitrary Motion of the Surface

Except in the case of small amplitude oscillations, we restricted our analysis to a patchwork of uniform motions of the surface. In this section we examine the general aspects of the solution of the problem when the motion is not limited to constant speed. The solution obtained in this case will be in the form of quadratures, which in general could only be used as the basis for a numerical calculation.

We assume that the radius of the sphere is specified as a function of time. Let this function be R(t). Only speeds less than c are allowed, so

$$|\dot{R}| < c \, . \tag{B.1}$$

Define

$$\eta = t - R(t)/c \tag{B.2}$$

and let the inverse relation be

$$t = S(\eta) \, . \tag{B.3}$$

The condition (B.1) ensures the existence of this function.

Applying the boundary condition $B_r = 0$ at $r = R$ we obtain the differential equation

$$f'(\eta) + \frac{c}{R} f(\eta) = cR^2 \tag{B.4}$$

which has appeared several times before.

But

$$R = c(t - \eta) = C(S(\eta) - \eta)$$

so that (B. 4) becomes

$$f'(\eta) + \frac{1}{S(\eta) - \eta} f(\eta) = c^3 (S(\eta) - \eta)^2 . \tag{B.5}$$

The solution of this equation is only a question of quadratures. In general the solution will be of the form

$$f(\eta) = \left(\exp \left[- \int d\eta/(S-\eta) \right] \right) \int c^3 (S-\eta)^2 \exp \left[\int d\eta/(S-\eta) \right] d\eta . \tag{B.6}$$

The main difficulty is therefore in finding $S(\eta)$. In the case of motions at a uniform speed this task is easy, and in addition the quadratures that appear in (B. 6) are also simple. For other types of motion, it is difficult or impossible to find analytical expressions for $S(\eta)$, and the quadratures are correspondingly more complicated.

One way in which other motions could be considered is to start with a function $S(\eta)$ for which all the quadratures can be done, and the only numerical work would then be to find $R(t)$. This will not be pursued any further in this paper, since it is doubtful that the functions $R(t)$ would be less artificial than the ones employed.